Riemann zeta function

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2013-03-21 14:04:51

1 Definition

The Riemann zeta function is defined to be the complex valued function given by the series

\[ \zeta(s) := \sum_{n=1}^{\infty} \frac{1}{n^s}, \]  

which is valid (in fact, absolutely convergent) for all complex numbers \( s \) with \( \text{Re}(s) > 1 \). We list here some of the key properties of the zeta function.

1. For all \( s \) with \( \text{Re}(s) > 1 \), the zeta function satisfies the Euler product formula

\[ \zeta(s) = \prod_p \frac{1}{1 - p^{-s}}, \]  

where the product is taken over all positive integer primes \( p \), and converges uniformly in a neighborhood of \( s \).

2. The zeta function has a meromorphic continuation to the entire complex plane with a simple pole at \( s = 1 \), of residue 1, and no other singularities.

3. The zeta function satisfies the functional equation

\[ \zeta(s) = 2^s \pi^{s-1} \sin \frac{\pi s}{2} \Gamma(1-s) \zeta(1-s), \]  

for any \( s \in \mathbb{C} \) (where \( \Gamma \) denotes the Gamma function).
2 Distribution of primes

The Euler product formula (??) given above expresses the zeta function as a product over the primes \( p \in \mathbb{Z} \), and consequently provides a link between the analytic properties of the zeta function and the distribution of primes in the integers. As the simplest possible illustration of this link, we show how the properties of the zeta function given above can be used to prove that there are infinitely many primes.

If the set \( S \) of primes in \( \mathbb{Z} \) were finite, then the Euler product formula
\[
\zeta(s) = \prod_{p \in S} \frac{1}{1 - p^{-s}}
\]
would be a finite product, and consequently \( \lim_{s \to 1} \zeta(s) \) would exist and would equal
\[
\lim_{s \to 1} \zeta(s) = \prod_{p \in S} \frac{1}{1 - p^{-1}}.
\]
But the existence of this limit contradicts the fact that \( \zeta(s) \) has a pole at \( s = 1 \), so the set \( S \) of primes cannot be finite.

A more sophisticated analysis of the zeta function along these lines can be used to prove both the analytic prime number theorem and Dirichlet’s theorem on primes in arithmetic progressions \(^1\). Proofs of the prime number theorem can be found in [? | ] and [? | ], and for proofs of Dirichlet’s theorem on primes in arithmetic progressions the reader may look in [? | ] and [? | ].

3 Zeros of the zeta function

A nontrivial zero of the Riemann zeta function is defined to be a root \( \zeta(s) = 0 \) of the zeta function with the property that \( 0 \leq \text{Re}(s) \leq 1 \). Any other zero is called trivial zero of the zeta function.

The reason behind the terminology is as follows. For complex numbers \( s \) with real part greater than 1, the series definition (??) immediately shows that no zeros of the zeta function exist in this region. It is then an easy matter to use the functional equation (??) to find all zeros of the zeta function with real part less than 0 (it turns out they are exactly the values \( -2n \), for \( n \) a positive integer). However, for values of \( s \) with real part between 0 and 1, the situation is quite different, since we have neither a series definition nor a functional equation to fall back upon; and indeed to this day very little is known about the behavior of the zeta function inside this critical strip of the complex plane.

It is known that the prime number theorem is equivalent to the assertion that the zeta function has no zeros \( s \) with \( \text{Re}(s) = 0 \) or \( \text{Re}(s) = 1 \). The celebrated Riemann hypothesis asserts that all nontrivial zeros \( s \) of the zeta function satisfy

\(^1\)In the case of arithmetic progressions, one also needs to examine the closely related Dirichlet \( L \)-functions in addition to the zeta function itself.
the much more precise equation $\text{Re}(s) = 1/2$. If true, the hypothesis would have profound consequences on the distribution of primes in the integers [?].
References


