three theorems on parabolas*

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In the Cartesian plane, pick a point with coordinates $(0, 2f)$ (subtle hint!) and construct (1) the set S of segments s joining $F = (0, 2f)$ with the points $(x, 0)$, and (2) the set B of right-bisectors b of the segments $s \in S$.

Theorem 2 : The *envelope* described by the lines of the set B is a parabola with x-axis as directrix and focal length $|f|$.

Proof: We're lucky in that we don't need a fancy definition of envelope; considering a line to be a set of points it's just the boundary of the set $C = \cup_{b \in B} b$. Strategy: fix an x coordinate and find the max/minimum of possible y's in C with that x. But first we'll pick an s from S by picking a point $p = (w, 0)$ on the x axis. The midpoint of the segment $s \in S$ through p is $M = (\frac{w}{2}, f)$. Also, the slope of this s is $-\frac{2f}{w}$. The corresponding right-bisector will also pass through $(\frac{w}{2}, f)$ and will have slope $\frac{w}{2f}$. Its equation is therefore

$$\frac{2y - 2f}{2x - w} = \frac{w}{2f}.$$

Equivalently,

$$y = f + \frac{wx}{2f} - \frac{w^2}{4f}.$$

By any of many very famous theorems (Euclid book II theorem twenty-something, Cauchy-Schwarz-Bunyakovski (overkill), differential calculus, what you will) for fixed x, y is an extremum for $w = x$ only, and therefore the envelope has equation

$$y = f + \frac{x^2}{4f}.$$

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I could say I'm done right now because we "know" that this is a parabola, with focal length f and x-axis as directrix. I don't want to, though. The most popular definition of parabola I know of is "set of points equidistant from some line d and some point f." The line responsible for the point on the envelope with given ordinate x was found to bisect the segment $s \in S$ through $H = (x, 0)$. So pick an extra point $Q \in b \in B$ where b is the perpendicular bisector of s. We then have $\angle FMQ = \angle QMH$ because they're both right angles, lengths $FM = MH$, and QM is common to both triangles FMQ and HMQ. Therefore two sides and the angles they contain are respectively equal in the triangles FMQ and HMQ, and so respective angles and respective sides are all equal. In particular, $FQ = QH$. Also, since Q and H have the same x coordinate, the line QH is the perpendicular to the x-axis, and so Q, a general point on the envelope, is equidistant from F and the x-axis. Therefore etc.

QED.

Because of this construction, it is clear that the lines of B are all tangent to the parabola in question.

We're not done yet. Pick a random point P outside C ("inside" the parabola), and call the parabola π (just to be nasty). Here's a nice quicky:

Theorem 3 The Reflector Law: For $R \in \pi$, the length of the path PRF is minimal when PR produced is perpendicular to the x-axis.

Proof: Quite simply, assume PR produced is not necessarily perpendicular to the x-axis. Because π *is* a parabola, the segment from R perpendicular to the x-axis has the same length as RF. So let this perpendicular hit the x-axis at H. We then have that the length of PRH equals that of PRF. But PRH (and hence PRF) is minimal when it's a straight line; that is, when PR produced is perpendicular to the x-axis.

QED

Hey! I called that theorem the "reflector law". Perhaps it didn't look like one. (It *is* in the Lagrangian formulation), but it's fairly easy to show (it's a similar argument) that the shortest path from a point to a line to a point makes "incident" and "reflected" angles equal.

One last marvelous tidbit. This will take more time, though. Let b be tangent to π at R, and let n be perpendicular to b at R. We will call n the *normal to* π *at* R. Let n meet the x-axis at G.

Theorem 4 : The radius of the "best-fit circle" to π at R is twice the length RG.

Proof: (Note: the $\approx$'s need to be phrased in terms of upper and lower bounds, so I can use the sandwich theorem, but the proof schema is exactly what is required).

Take two points R, R' on π some small distance ϵ from each other (we don't actually use ϵ, it's just a psychological trick). Construct the tangent t and normal n at R, normal n' at R'. Let n, n' intersect at O, and t intersect the x-axis at G. Join $RF, R'F$. Erect perpendiculars g, g' to the x-axis through R, R' respectively. Join RR'. Let g intersect the x-axis at H. Let P, P' be points on g, g' not in C. Construct RE perpendicular to RF with E in $R'F$. We now have

i) $\angle PRO = \angle ORF = \angle GRH \approx \angle P'R'O = \angle OR'F$

ii) $ER \approx FR \cdot \angle EFR$

iii) $\angle R'RE + \angle ERO \approx \frac{\pi}{2}$ (That's the number π, not the parabola)

iv) $\angle ERO + \angle ORF = \frac{\pi}{2}$

v) $\angle R'ER \approx \frac{\pi}{2}$

vi) $\angle R'OR = \frac{1}{2}\angle R'FR$

vii) $R'R \approx OR \cdot \angle R'OR$

viii) $FR = RH$

From (iii),(iv) and (i) we have $\angle R'RE \approx \angle GRH$, and since R' is close to R, and if we let R' approach R, the approximations approach equality. Therefore, we have that triangle $R'RE$ approaches similarity with GRH. Therefore we have $RR' : ER \approx RG : RH$. Combining this with (ii),(vi),(vii), and (viii) it follows that $RO \approx 2RG$, and in the limit $R' \to R$, $RO = 2RG$.

QED

This last theorem is a very nice way of short-cutting all the messy calculus needed to derive the Schwarzschild "Black-Hole" solution to Einstein's field equations, and that's why I enjoy it so.