proof of Schwarz lemma*

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Define $g(z) = f(z)/z$. Then $g : \Delta \to \mathbb{C}$ is a holomorphic function. The Schwarz lemma is just an application of the maximal modulus principle to g.

For any $1 > \epsilon > 0$, by the maximal modulus principle $|g|$ must attain its maximum on the closed disk $\{z : |z| \leq 1 - \epsilon\}$ at its boundary $\{z : |z| = 1 - \epsilon\}$, say at some point z_ϵ. But then $|g(z)| \leq |g(z_\epsilon)| \leq \frac{1}{1-\epsilon}$ for any $|z| \leq 1 - \epsilon$. Taking an infinimum as $\epsilon \to 0$, we see that values of g are bounded: $|g(z)| \leq 1$.

Thus $|f(z)| \leq |z|$. Additionally, $f'(0) = g(0)$, so we see that $|f'(0)| = |g(0)| \leq 1$. This is the first part of the lemma.

Now suppose, as per the premise of the second part of the lemma, that $|g(w)| = 1$ for some $w \in \Delta$. For any $r > |w|$, it must be that $|g|$ attains its maximal modulus (1) *inside* the disk $\{z : |z| \leq r\}$, and it follows that g must be constant inside the entire open disk Δ. So $g(z) \equiv a$ for $a = g(w)$ of modulus 1, and $f(z) = az$, as required.

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