circular helix*

rspuzio[†]

2013-03-21 15:31:59

The space curve traced out by the parameterization

$$\boldsymbol{\gamma}(t)=\begin{bmatrix} a\cos(t)\\ a\sin(t)\\ bt \end{bmatrix},\quad t\in\mathbb{R},\ a,b\in\mathbb{R}$$

is called a *circular helix* (plur. *helices*).

Its Frenet frame is:

$$\mathbf{T}=\frac{1}{\sqrt{a^2+b^2}}\begin{bmatrix} -a\sin t\\ a\cos t\\ b \end{bmatrix},$$
$$\mathbf{N}=\begin{bmatrix} -\cos t\\ -\sin t\\ 0 \end{bmatrix},$$
$$\mathbf{B}=\frac{1}{\sqrt{a^2+b^2}}\begin{bmatrix} b\sin t\\ -b\cos t\\ a \end{bmatrix}.$$

Its curvature and torsion are the following constants:

$$\kappa=\frac{a}{a^2+b^2},\quad \tau=\frac{b}{a^2+b^2}.$$

A circular helix can be conceived of as a space curve with constant, non-zero curvature, and constant, non-zero torsion. Indeed, one can show that if a space curve satisfies the above constraints, then there exists a system of Cartesian coordinates in which the curve has a parameterization of the form shown above.

An important property of the circular helix is that for any point of it, the angle φ between its tangent and the helix axis is constant. Indeed, if we consider

*⟨*CircularHelix*⟩ created: ⟨*2013-03-21*⟩ by: ⟨*rspuzio*⟩ version: ⟨*33928*⟩ Privacy setting: ⟨*1*⟩ ⟨*Definition*⟩ ⟨*53A04*⟩

Figure 1: A plot of a circular helix with $a = b = 1$, and $\kappa = \tau = 1/2$.

the position vector of that arbitrary point, we have (where $\mathbf{k}$ is the unit vector parallel to helix axis)

$$\frac{d\boldsymbol{\gamma}}{dt} \cdot \mathbf{k} = \begin{bmatrix} -a \sin t \\ a \cos t \\ b \end{bmatrix} \begin{bmatrix} 0 \ 0 \ 1 \end{bmatrix} = b \equiv \left\| \frac{d\boldsymbol{\gamma}}{dt} \right\| \cos\varphi = \sqrt{a^2 + b^2} \cos\varphi.$$

Therefore,

$$\cos\varphi = \frac{b}{\sqrt{a^2 + b^2}} \text{constant},$$

as was to be shown.

There is also another parameter, the so-called *pitch of the helix* P which is the separation between two consecutive turns. (It is mostly used in the manufacture of screws.) Thus,

$$P = \gamma_3(t + 2\pi) - \gamma_3(t) = b(t + 2\pi) - bt = 2\pi b\,,$$

and P is also a constant.