units of quadratic fields*

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Dirichlet's unit theorem gives all units of an algebraic number field $\mathbb{Q}(\vartheta)$ in the unique form

$$\varepsilon = \zeta^n \eta_1^{k_1} \eta_2^{k_2} ... \eta_t^{k_t},$$

where ζ is a primitive w^{th} root of unity in $\mathbb{Q}(\vartheta)$, the η_j's are the fundamental units of $\mathbb{Q}(\vartheta)$, $0 \leqq n \leqq w-1$, $k_j \in \mathbb{Z}\ \forall j$, $t = r+s-1$.

- The case of a real quadratic field $\mathbb{Q}(\sqrt{m})$, the square-free $m > 1$: $r = 2$, $s = 0$, $t = r+s-1 = 1$. So we obtain

  $$\varepsilon = \zeta^n \eta^k = \pm \eta^k,$$

  because $\zeta = -1$ is the only real primitive root of unity ($w = 2$). Thus, every real quadratic field has infinitely many units and a unique fundamental unit η.

  Examples: If $m = 3$, then $\eta = 2+\sqrt{3}$; if $m = 421$, then $\eta = \frac{444939+21685\sqrt{421}}{2}$.

- The case of any imaginary quadratic field $\mathbb{Q}(\vartheta)$; here $\vartheta = \sqrt{m}$, the square-free $m < 0$: The conjugates of ϑ are the pure imaginary numbers $\pm\sqrt{m}$, hence $r = 0$, $2s = 2$, $t = r+s-1 = 0$. Thus we see that all units are

  $$\varepsilon = \zeta^n.$$

  1) $m = -1$. The field contains the primitive fourth root of unity, e.g. i, and therefore all units in the *Gaussian field* $\mathbb{Q}(i)$ are i^n, where $n = 0, 1, 2, 3$.

  2) $m = -3$. The field in question is a cyclotomic field containing the primitive third root of unity and also the primitive sixth root of unity, namely

  $$\zeta = \cos\frac{2\pi}{6} + i\sin\frac{2\pi}{6};$$

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hence all units are $\varepsilon = (\frac{1+\sqrt{-3}}{2})^n$, where $n = 0, 1, \ldots, 5$, or, equivalently, $\varepsilon = \pm(\frac{-1+\sqrt{-3}}{2})^n$, where $n = 0, 1, 2$.

3) $m = -2$, $m < -3$. The only roots of unity in the field are ± 1; hence $\zeta = -1$, $w = 2$, and the units of the field are simply $(-1)^n$, where $n = 0, 1$.