Weierstrass factorization theorem*

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There are several different statements of this theorem, but in essence this theorem will allow us to prescribe zeros and their orders of a holomorphic function. It also allows us to factor any holomorphic function into a product of zeros and a non-zero holomorphic function. We will need to know here how an infinite product converges. It can then be shown that if $\prod_{k=1}^{\infty} f_k(z)$ converges uniformly and absolutely on compact subsets, then it converges to a holomorphic function given that all the $f_k(z)$ are holomorphic. This is what we will mean by the infinite product in what follows.

Note that once we can prescribe zeros of a function then we can also prescribe the poles as well and get a meromorphic function just by dividing two holomorphic functions f/h where f will contribute zeros, and h will make poles at the points where $h(z) = 0$. So let's start with the existence statement.

Theorem (Weierstrass Product)**.** *Let $G \subset \mathbb{C}$ be a domain, let $\{a_k\}$ be a sequence of points in G with no accumulation points in G, and let $\{n_k\}$ be any sequence of non-zero integers (positive or negative). Then there exists a function f meromorphic in G whose poles and zeros are exactly at the points a_k and the order of the pole or zero at a_k is n_k (a positive order stands for zero, negative stands for pole).*

Next let's look at a more specific statement with more restrictions. For one let's start looking at the whole complex plane and further let's forget about poles for now to make the following formulas simpler.

Definition. We call

$$E_0(z) := 1 - z,$$
$$E_p(z) := (1 - z)e^{z+\frac{1}{2}z^2+\cdots+\frac{1}{p}z^p} \qquad \text{for } p \geq 1,$$

an *elementary factor*.

Now note that for some $a \in \mathbb{C}\backslash\{0\}$, $E_p(z/a)$ has a simple zero (zero of order 1) at a.

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Theorem (Weierstrass Factorization)**.** *Suppose f be an entire function and let $\{a_k\}$ be the zeros of f such that $a_k \neq 0$ (the non-zero zeros of f). Let m be the order of the zero of f at $z = 0$ ($m = 0$ if f does not have a zero at $z = 0$). Then there exists an entire function g and a sequence of non-negative integers $\{p_k\}$ such that*

$$f(z) = z^m e^{g(z)} \prod_{k=1}^{\infty} E_{p_k}\left(\frac{z}{a_k}\right).$$

Note that we can always choose $p_k = k - 1$ and the product above will converge as needed, but we may be able to choose better p_k for specific functions.

Example. *As an example we can try to factorize the function $\sin(\pi z)$, which has zeros at all the integers. Applying the Weierstrass factorization theorem directly we get that*

$$\sin(\pi z) = z e^{g(z)} \prod_{k=-\infty, k\neq 0}^{\infty} \left(1 - \frac{z}{k}\right) e^{z/k},$$

where $g(z)$ is some holomorphic function. It turns out that $e^{g(z)} = \pi$, and rearranging the product we get

$$\sin(\pi z) = z\pi \prod_{k=1}^{\infty} \left(1 - \frac{z^2}{k^2}\right).$$

*This is an example where we could choose the $p_k = 1$ for all k and thus we could then get rid of the ugly parts of the infinite product. For complete calculations in this example see Conway [**?**].*

References

[1] John B. Conway. *Functions of One Complex Variable I.* Springer-Verlag, New York, New York, 1978.

[2] Theodore B. Gamelin. *Complex Analysis.* Springer-Verlag, New York, New York, 2001.