

# generators of inverse ideal\*

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**Theorem.** Let  $R$  be a commutative ring with non-zero unity and let  $T$  be the total ring of fractions of  $R$ . If  $\mathfrak{a} = (a_1, \dots, a_n)$  is an invertible *fractional ideal* of  $R$  with  $\mathfrak{a}\mathfrak{b} = R$ , then also the inverse ideal  $\mathfrak{b}$  can be generated by  $n$  elements of  $T$ .

*Proof.* The equation  $\mathfrak{a}\mathfrak{b} = (1)$  implies the existence of the elements  $a'_i$  of  $\mathfrak{a}$  and  $b'_i$  of  $\mathfrak{b}$  ( $i = 1, \dots, m$ ) such that  $a'_1 b'_1 + \dots + a'_m b'_m = 1$ . Because the  $a'_i$ 's are in  $\mathfrak{a}$ , they may be expressed as

$$a'_i = \sum_{j=1}^n r_{ij} a_j \quad (i = 1, \dots, m),$$

where the  $r_{ij}$ 's are some elements of  $R$ . Now the unity acquires the form

$$1 = \sum_{i=1}^m a'_i b'_i = \sum_{i=1}^m \sum_{j=1}^n r_{ij} a_j b'_i = \sum_{j=1}^n a_j \sum_{i=1}^m r_{ij} b'_i = \sum_{j=1}^n a_j b_j,$$

in which

$$b_j = \sum_{i=1}^m r_{ij} b'_i \in R\mathfrak{b} = \mathfrak{b} \quad (j = 1, \dots, n).$$

Thus an arbitrary element  $b$  of the fractional ideal  $\mathfrak{b}$  satisfies the condition

$$b = b \cdot 1 = \sum_{j=1}^n (a_j b) b_j \in Rb_1 + \dots + Rb_n = (b_1, \dots, b_n).$$

Consequently,  $\mathfrak{b} \subseteq (b_1, \dots, b_n)$ . Since the inverse inclusion is apparent, we have the equality

$$\mathfrak{a}^{-1} = \mathfrak{b} = (b_1, \dots, b_n).$$

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