Recall that the symmetric difference of two sets $A, B$ is the set $A \cup B - (A \cap B)$. In this entry, we list and prove some of the basic properties of $\triangle$.

1. (commutativity of $\triangle$) $A \triangle B = B \triangle A$, because $\cup$ and $\cap$ are commutative.

2. If $A \subseteq B$, then $A \triangle B = B - A$, because $A \cup B = B$ and $A \cap B = A$.

3. $A \triangle \emptyset = A$, because $\emptyset \subseteq A$, and $A - \emptyset = A$.

4. $A \triangle A = \emptyset$, because $A \subseteq A$ and $A - A = \emptyset$.

5. $A \triangle B = (A - B) \cup (B - A)$ (hence the name symmetric difference).

   Proof. $A \triangle B = (A \cup B) - (A \cap B) = (A \cup B) \cap (A \cap B)' = (A \cup B) \cap (A' \cup B') = (A \cup B) \cap (A' \cup B') = (B \cap A') \cup (A \cap B') = (B - A) \cup (A - B)$. □

6. $A' \triangle B' = A \triangle B$, because $A' \triangle B' = (A' - B') \cup (B' - A') = (A' \cap B) \cup (B' \cap A) = (B - A) \cap (A - B) = A \triangle B$.

7. (distributivity of $\cap$ over $\triangle$) $A \cap (B \triangle C) = (A \cap B) \triangle (A \cap C)$.

   Proof. $A \cap (B \triangle C) = A \cap ((B \cup C) - (B \cap C))$, which is $(A \cap (B \cup C)) - (A \cap (B \cap C))$, one of the properties of set difference (see proof [here]). This in turns is equal to $((A \cap B) \cup (A \cap C)) - ((A \cap B) \cap (A \cap C)) = (A \cap B) \triangle (A \cap C)$. □

8. (associativity of $\triangle$) $(A \triangle B) \triangle C = A \triangle (B \triangle C)$.

   Proof. Let $U$ be a set containing $A, B, C$ as subsets (take $U = A \cup B \cup C$ if necessary). For a given $B$, let $f : P(U) \times P(U) \rightarrow P(U)$ be a function defined by $f(A, C) = (A \triangle B) \triangle C$. Associativity of $\triangle$ is then then same...
as showing that $f(A, C) = f(C, A)$, since $A \triangle (B \triangle C) = (B \triangle C) \triangle A = (C \triangle B) \triangle A$.

By expanding $f(A, C)$, we have

\[
(A \triangle B) \triangle C = ((A \triangle B) - C) \cup (C - (A \triangle B))
\]

\[
= (((A \cap B') \cup (B \cap A')) \cap C') \cup ((C \cap A \cap B) \cup (C - (A \cup B)))
\]

\[
= ((A \cap B' \cap C') \cup (B \cap A' \cap C')) \cup ((C \cap A \cap B) \cup (C \cap A' \cap B'))
\]

\[
= (B \cap A' \cap C') \cup (B \cap A \cap C) \cup (B' \cap A \cap C') \cup (B' \cap A' \cap C).
\]

It is now easy to see that the last expression does not change if one exchanges $A$ and $C$. Hence, $f(A, C) = f(C, A)$ and this shows that $\triangle$ is associative.

\[\square\]

Remark. All of the properties of $\triangle$ on sets can be generalized to $\triangle$ on Boolean algebras.