complex exponential function*

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The *complex exponential function* $\exp : \mathbb{C} \to \mathbb{C}$ may be defined in many equivalent ways: Let $z = x+iy$ where $x,\, y \in \mathbb{R}$.

- $\exp z \;:=\; e^x(\cos y + i\sin y)$
- $\exp z \;:=\; \lim_{n\to\infty}\left(1+\frac{z}{n}\right)^n$
- $\exp z \;:=\; \sum_{n=0}^{\infty}\frac{z^n}{n!}$

The complex exponential function is usually denoted in power form:

$$e^z \;:=\; \exp z,$$

where e is the Napier's constant. It also coincincides with the real exponential function when z is real (choose $y = 0$). It has all the properties of power, e.g. $e^{-z} = \frac{1}{e^z}$; these are consequences of the addition formula

$$e^{z_1+z_2} \;=\; e^{z_1}e^{z_2}$$

of the complex exponential function.

The function gets all complex values except 0 and is periodic having the *prime period* (the period with least non-zero modulus) $2\pi i$. The exp is holomorphic, its derivative

$$\frac{d}{dz}e^z \;=\; e^z,$$

which is obtained from the series form via termwise differentiation, is similar as in $\mathbb{R}$.

So we have a fourth way to define

- $\exp z \;:=\; w(z)$

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with w the solution of the differential equation $\dfrac{dw}{dz} = w$ under the initial condition $w(0) = 1$.

Some formulae:

$$|e^z| = e^x, \quad \arg e^z = y + 2n\pi \quad (n = 0, \pm1, \pm2, \ldots),$$

$$\mathrm{Re}(e^z) = e^x \cos y, \quad \mathrm{Im}(e^z) = e^x \sin y$$