Two harmonic functions $u$ and $v$ from an open subset $A$ of $\mathbb{R} \times \mathbb{R}$ to $\mathbb{R}$, which satisfy the Cauchy-Riemann equations
\[ u_x = v_y, \quad u_y = -v_x, \tag{1} \]
are the harmonic conjugate functions of each other.

- The relationship between $u$ and $v$ has a simple geometric meaning: Let’s determine the slopes of the constant-value curves $u(x, y) = a$ and $v(x, y) = b$ in any point $(x, y)$ by differentiating these equations. The first gives $u_x dx + u_y dy = 0$, or
\[ \frac{dy}{dx}^{(u)} = -\frac{u_x}{u_y} = \tan \alpha, \]
and the second similarly
\[ \frac{dy}{dx}^{(v)} = -\frac{v_x}{v_y} \]
but this is, by virtue of (1), equal to
\[ \frac{u_y}{u_x} = -\frac{1}{\tan \alpha}. \]
Thus, by the condition of orthogonality, the curves intersect at right angles in every point.

- If one of $u$ and $v$ is known, then the other may be determined with (1): When e.g. the function $u$ is known, we need only to calculate the line integral
\[ v(x, y) = \int_{(x_0, y_0)}^{(x, y)} (-u_y \, dx + u_x \, dy) \]
along any path connecting $(x_0, y_0)$ and $(x, y)$ in $A$. The result is the harmonic conjugate $v$ of $u$, unique up to a real addend if $A$ is simply connected.
• It follows from the preceding, that every harmonic function has a harmonic conjugate function.

• The real part and the imaginary part of a holomorphic function are always the harmonic conjugate functions of each other.

**Example.**  $\sin x \cosh y$ and $\cos x \sinh y$ are harmonic conjugates of each other.