complex sine and cosine*

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We define for all complex values of z:

- $\sin z \;:=\; z-\frac{z^3}{3!}+\frac{z^5}{5!}-\frac{z^7}{7!}+-\ldots$
- $\cos z \;:=\; 1-\frac{z^2}{2!}+\frac{z^4}{4!}-\frac{z^6}{6!}+-\ldots$

Because these series converge for all real values of z, their radii of convergence are ∞, and therefore they converge for all complex values of z (by a known theorem of Abel; cf. the entry power series), too. Thus they define holomorphic functions in the whole complex plane, i.e. entire functions (to be more precise, entire transcendental functions). The series also show that sine is an odd function and cosine an even function.

Expanding the complex exponential functions e^{iz} and e^{-iz} to power series and separating the terms of even and odd degrees gives the generalized Euler's formulas

$$e^{iz} \;=\; \cos z+i\sin z, \quad e^{-iz} \;=\; \cos z-i\sin z.$$

Adding, subtracting and multiplying these two formulae give respectively the two Euler's formulae

$$\cos z \;=\; \frac{e^{iz}+e^{-iz}}{2}, \quad \sin z \;=\; \frac{e^{iz}-e^{-iz}}{2i} \tag{1}$$

(which sometimes are used to define cosine and sine) and the "fundamental formula of trigonometry"

$$\cos^2 z+\sin^2 z \;=\; 1.$$

As consequences of the generalized Euler's formulae one gets easily the addition formulae of sine and cosine:

$$\sin{(z_1+z_2)} \;=\; \sin z_1\cos z_2+\cos z_1\sin z_2,$$

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$$\cos(z_1+z_2) \;=\; \cos z_1 \cos z_2 - \sin z_1 \sin z_2;$$

so they are in $\mathbb{C}$ fully similar as in $\mathbb{R}$. It means that all *goniometric* formulae derived from these, such as

$$\sin 2z \;=\; 2\sin z \cos z, \quad \sin(\pi-z) \;=\; \sin z, \quad \sin^2 z \;=\; \frac{1-\cos 2z}{2},$$

have the old shape. See also the persistence of analytic relations.

The addition formulae may be written also as

$$\sin(x+iy) \;=\; \sin x \cosh y + i\cos x \sinh y,$$

$$\cos(x+iy) \;=\; \cos x \cosh y - i\sin x \sinh y$$

which imply, when assumed that $x,\, y \in \mathbb{R}$, the results

$$\mathrm{Re}(\sin(x+iy)) \;=\; \sin x \cosh y, \quad \mathrm{Im}(\sin(x+iy)) \;=\; \cos x \sinh y,$$

$$\mathrm{Re}(\cos(x+iy)) \;=\; \cos x \cosh y, \quad \mathrm{Im}(\cos(x+iy)) \;=\; -\sin x \sinh y.$$

Thus we get the modulus estimation

$$|\sin(x+iy)| \;=\; \sqrt{\sin^2 x \cosh^2 y + \cos^2 x \sinh^2 y} \;=\; \sqrt{\sin^2 x \cosh^2 y + (1-\sin^2 x)\sinh^2 y}$$
$$=\; \sqrt{\sin^2 x(\cosh^2 y - \sinh^2 y) + \sinh^2 y} \;=\; \sqrt{\sin^2 x \cdot 1 + \sinh^2 y} \;\geq\; |\sinh y|,$$

which tends to infinity when $z = x+iy$ moves to infinity along any line non-parallel to the real axis. The modulus of $\cos(x+iy)$ behaves similarly.

Another important consequence of the addition formulae is that the functions sin and cos are periodic and have 2π as their prime period:

$$\sin(z+2\pi) \;=\; \sin z, \quad \cos(z+2\pi) \;=\; \cos z \quad \forall z$$

The periodicity of the functions causes that their inverse functions, the *complex cyclometric functions*, are infinitely multivalued; they can be expressed via the complex logarithm and square root (see general power) as

$$\arcsin z \;=\; \frac{1}{i}\log(iz+\sqrt{1-z^2}), \quad \arccos z \;=\; \frac{1}{i}\log(z+i\sqrt{1-z^2}).$$

The derivatives of sine function and cosine function are obtained either from the series forms or from (1):

$$\frac{d}{dz}\sin z \;=\; \cos z, \quad \frac{d}{dz}\cos z \;=\; -\sin z$$

Cf. the higher derivatives.