positive cone*

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Let R be a commutative ring with 1. A subset P of R is called a *pre-positive cone* of R provided that

1. $P+P\subseteq P$ (P is additively closed)
2. $P\cdot P\subseteq P$ (P is multiplicatively closed)
3. $-1\notin P$
4. $\operatorname{sqr}(R):=\{r^2\mid r\in R\}\subseteq P$.

As it turns out, a field endowed with a pre-positive cone has an order structure. The field is called a formally real, orderable, or ordered field. Before defining what this "order" is, let's do some preliminary work. Let P_0 be a pre-positive cone of a field F. By Zorn's Lemma, the set of pre-positive cones extending P_0 has a maximal element P. It can be shown that P has two additional properties:

5. $P\cup(-P)=F$
6. $P\cap(-P)=(0)$.

Proof. First, suppose there is $a\in F-(P\cup(-P))$. Let $\overline{P}=P+Pa$. Then $a\in\overline{P}$ and so P is strictly contained in $\overline{P}$. Clearly, $\operatorname{sqr}(F)\subseteq\overline{P}$ and $\overline{P}$ is easily seen to be additively closed. Also, $\overline{P}$ is multiplicatively closed as the equation $(p_1+q_1a)(p_2+q_2a)=(p_1p_2+q_1q_2a^2)+(p_1q_2+q_1p_2)a$ demonstrates. Since P is a maximal and $\overline{P}$ properly contains P, $\overline{P}$ is not a pre-positive cone, which means $-1\in\overline{P}$. Write $-1=p+qa$. Then $q(-a)=p+1\in P$. Since $q\in P$, $1/q=q(1/q)^2\in P$, $-a=(1/q)(p+1)\in P$, contradicting the assumption that $a\notin -P$. Therefore, $P\cup(-P)=F$.

For the second part, suppose $a\in P\cap(-P)$. Since $a\in -P$, $-a\in P$. If $a\neq 0$, then $-1=a(-a)(1/a)^2\in P$, a contradiction. □

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A subset P of a field F satisfying conditions 1, 2, 5 and 6 is called a *positive cone* of F. A positive cone is a pre-positive cone. If $a \in F$, then either $a \in P$ or $-a \in P$. In either case, $a^2 \in P$. Next, if $-1 \in P$, then $1 \in -P$. But $1 = 1^2 \in P$, we have $1 \in P \cap (-P)$, contradicting Condition 6 of P.

Now, define a binary relation $\leq$, on F by:

$$a \leq b \Longleftrightarrow b - a \in P$$

It is not hard to see that $\leq$ is a total order on F. In addition, with the additive and multiplicative structures on F, we also have the following two rules:

1. $a \leq b \Rightarrow a + c \leq b + c$
2. $0 \leq a$ and $0 \leq b \Rightarrow 0 \leq ab$.

Thus, F is a field ordered by $\leq$.

Remark. Positive cones may be defined for more general ordered algebraic structures, such as partially ordered groups, or partially ordered rings.

References

[1] A. Prestel, *Lectures on Formally Real Fields*, Springer, 1984