

non-isomorphic completions of \mathbb{Q}^*

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No field \mathbb{Q}_p of the *p-adic numbers* (*p*-adic rationals) is isomorphic with the field \mathbb{R} of the real numbers.

Proof. Let's assume the existence of a field isomorphism $f : \mathbb{R} \rightarrow \mathbb{Q}_p$ for some positive prime number p . If we denote $f(\sqrt{p}) = a$, then we obtain

$$a^2 = (f(\sqrt{p}))^2 = f((\sqrt{p})^2) = f(p) = p,$$

because the isomorphism maps the elements of the prime subfield on themselves. Thus, if $|\cdot|_p$ is the normed *p*-adic valuation of \mathbb{Q} and of \mathbb{Q}_p , we get

$$|a|_p = \sqrt{|a^2|_p} = \sqrt{|p|_p} = \sqrt{\frac{1}{p}},$$

which value is an irrational number as a square root of a non-square rational. But this is impossible, since the value group of the completion \mathbb{Q}_p must be the same as the value group $|\mathbb{Q} \setminus \{0\}|_p$ which consists of all integer powers of p . So we conclude that there can not exist such an isomorphism.

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