non-isomorphic completions of $\mathbb{Q}$*

pahio†

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No field $\mathbb{Q}_p$ of the *p-adic numbers* (p-adic rationals) is isomorphic with the field $\mathbb{R}$ of the real numbers.

Proof. Let's assume the existence of a field isomorphism $f:\,\mathbb{R}\to\mathbb{Q}_p$ for some positive prime number p. If we denote $f(\sqrt{p})=a,$ then we obtain

$$a^2=(f(\sqrt{p}))^2=f((\sqrt{p})^2)=f(p)=p,$$

because the isomorphism maps the elements of the prime subfield on themselves. Thus, if $|\cdot|_p$ is the normed p-adic valuation of $\mathbb{Q}$ and of $\mathbb{Q}_p$, we get

$$|a|_p=\sqrt{|a^2|_p}=\sqrt{|p|_p}=\sqrt{\frac{1}{p}},$$

which value is an irrational number as a square root of a non-square rational. But this is impossible, since the value group of the completion $\mathbb{Q}_p$ must be the same as the value group $|\mathbb{Q}\setminus\{0\}|_p$ which consists of all integer powers of p. So we conclude that there can not exist such an isomorphism.

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