# theorems on sums of squares*

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**Theorem** (**Hurwitz Theorem**). *Let $F$ be a field with characteristic not 2. The sum of squares identity of the form*

$$(x_1^2+\cdots+x_n^2)(y_1^2+\cdots+y_n^2)=z_1^2+\cdots+z_n^2$$

*where each $z_k$ is bilinear over $x_i$ and $y_j$ (with coefficients in $F$), is possible iff $n=1,2,4,8$.*

**Remarks**.

1. When the ground field is $\mathbb{R}$, this theorem is equivalent to the fact that the only normed real division alternative algebra is one of $\mathbb{R}$, $\mathbb{C}$, $\mathbb{H}$, $\mathbb{O}$, as one observes that the sums of squares can be interpreted as the square of the norm defined for each of the above algebras.
2. An equivalent characterization is that the above four mentioned algebras are the only real composition algebras.

A generalization of the above is the following:

**Theorem** (**Pfister's Theorem**). *Let $F$ be a field of characteristic not 2. The sum of squares identity of the form*

$$(x_1^2+\cdots+x_n^2)(y_1^2+\cdots+y_n^2)=z_1^2+\cdots+z_n^2$$

*where each $z_k$ is a rational function of $x_i$ and $y_j$ (element of $F(x_1,\dots,x_n,y_1,\dots,y_n)$), is possible iff $n$ is a power of 2.*

**Remark**. The form of Pfister's theorem is stated in a way so as to mirror the form of Hurwitz theorem. In fact, Pfister proved the following: if $F$ is a field and $n$ is a power of 2, then there exists a sum of squares identity of the form

$$(x_1^2+\cdots+x_n^2)(y_1^2+\cdots+y_n^2)=z_1^2+\cdots+z_n^2$$

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such that each $z_k$ is a rational function of the $x_i$ and a linear function of the $y_j$, or that

$$z_k = \sum_{j=1}^{n} r_{kj} y_j \qquad \text{where } r_{kj} \in F(x_1, \ldots, x_n).$$

Conversely, if $n$ is not a power of 2, then there exists a field $F$ such that the above sum of square identity does not hold for *any* $z_i \in F(x_1, \ldots, x_n, y_1, \ldots, y_n)$. Notice that $z_i$ is no longer required to be a linear function of the $y_j$ anymore.

When $F$ is the field of reals $\mathbb{R}$, we have the following generalization, also due to Pfister:

**Theorem.** *If $f \in \mathbb{R}(X_1, \ldots, X_n)$ is positive semidefinite, then $f$ can be written as a sum of $2^n$ squares.*

The above theorem is very closely related to Hilbert's 17th Problem:

**Hilbert's 17th Problem.** *Whether it is possible, to write a positive semidefinite rational function in $n$ indeterminates over the reals, as a sum of squares of rational functions in $n$ indeterminates over the reals?*

The answer is yes, and it was proved by Emil Artin in 1927. Additionally, Artin showed that the answer is also yes if the reals were replaced by the rationals.

# References


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