

# invariance of formula for surface integration with respect to area under change of variables\*

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First, we can use the chain rule for Jacobians to see how one of the terms in parentheses transforms:

$$\frac{\partial(x, y)}{\partial(u, v)} = \frac{\partial(x, y)}{\partial(u', v')} \frac{\partial(u', v')}{\partial(u, v)}$$

A similar story holds for the other two factors. Combining them, we conclude that

$$\begin{aligned} & \sqrt{\left(\frac{\partial(x, y)}{\partial(u, v)}\right)^2 + \left(\frac{\partial(y, z)}{\partial(u, v)}\right)^2 + \left(\frac{\partial(z, x)}{\partial(u, v)}\right)^2} = \\ & \sqrt{\left(\frac{\partial(x, y)}{\partial(u', v')} \frac{\partial(u', v')}{\partial(u, v)}\right)^2 + \left(\frac{\partial(y, z)}{\partial(u', v')} \frac{\partial(u', v')}{\partial(u, v)}\right)^2 + \left(\frac{\partial(z, x)}{\partial(u', v')} \frac{\partial(u', v')}{\partial(u, v)}\right)^2} = \\ & \frac{\partial(u', v')}{\partial(u, v)} \sqrt{\left(\frac{\partial(x, y)}{\partial(u', v')}\right)^2 + \left(\frac{\partial(y, z)}{\partial(u', v')}\right)^2 + \left(\frac{\partial(z, x)}{\partial(u', v')}\right)^2} \end{aligned}$$

Since the factor in parentheses in front of the square root is the Jacobi determinant, we can apply the rule change of variables in multidimensional integrals to conclude that

$$\begin{aligned} & \int f(u, v) \sqrt{\left(\frac{\partial(x, y)}{\partial(u, v)}\right)^2 + \left(\frac{\partial(y, z)}{\partial(u, v)}\right)^2 + \left(\frac{\partial(z, x)}{\partial(u, v)}\right)^2} du dv = \\ & \int f(u', v') \sqrt{\left(\frac{\partial(x, y)}{\partial(u', v')}\right)^2 + \left(\frac{\partial(y, z)}{\partial(u', v')}\right)^2 + \left(\frac{\partial(z, x)}{\partial(u', v')}\right)^2} du' dv', \end{aligned}$$

which shows that our formula gives the same answer for  $\int_S f(u, v) d^2A$ , no matter how we choose to parameterize  $S$ .

\**(InvarianceOfFormulaForSurfaceIntegrationWithRespectToAreaUnderChangeOfVariables)*  
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