regular elements of finite ring*

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2013-03-21 19:02:17

Theorem. If the finite ring R has regular elements, then it has a unity. All regular elements of R form a group under the ring multiplication and with identity element the unity of R. Thus the regular elements are exactly the units of the ring; the rest of the elements are the zero and the zero divisors.

Proof. Obviously, the set of the regular elements is non-empty and closed under the multiplication. Let's think the multiplication table of this set. It is a finite square where every row only contains distinct elements (any equation $ax = ay$ reduces to $x = y$). Hence, for every regular element a, the square a^2 determines another a' such that $a^2a' = a$. This implies $a'(a^2a')(aa') = a'a(aa')$, i.e. $(a'a)(aa')^2 = (a'a)(aa')$, and since $a'a$ is regular, we obtain that $(aa')^2 = aa'$. So aa' is idempotent, and because it also is regular, it must be the unity of the ring: $aa' = 1$. Thus we see that R has a unity which is a regular element and that a has a multiplicative inverse a', also regular. Consequently the regular elements form a group.

*⟨RegularElementsOfFiniteRing⟩ created: ⟨2013-03-21⟩ by: ⟨pahio⟩ version: ⟨36941⟩ Privacy setting: ⟨1⟩ ⟨Theorem⟩ ⟨13G05⟩ ⟨16U60⟩