derivation of 2D reflection matrix*

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Reflection across a line of given angle

Let $\mathbf{x}, \mathbf{y}$ be perpendicular unit vectors in the plane. Suppose we want to reflect vectors (perpendicularly) over a line that makes an angle θ with the positive $\mathbf{x}$ axis. More precisely, we are given a direction direction vector $\mathbf{u} = \cos\theta\,\mathbf{x} + \sin\theta\,\mathbf{y}$ for the line of reflection. A unit vector perpendicular to $\mathbf{u}$ is $\mathbf{v} = -\sin\theta\,\mathbf{x} + \cos\theta\,\mathbf{y}$ (as is easily checked). Then to reflect an arbitrary vector $\mathbf{w}$, we write $\mathbf{w}$ in terms of its components in the $\mathbf{u}, \mathbf{v}$ axes: $\mathbf{w} = a\mathbf{u} + b\mathbf{v}$, and the result of the reflection is to be $\mathbf{w}' = a\mathbf{u} - b\mathbf{v}$.

We compute the matrix for such a reflection in the original x, y coordinates.

Denote the reflection by T. By the matrix change-of-coordinates formula, we have

$$[T]_{xy} = [I]^{xy}_{uv}\,[T]_{uv}\,[I]^{uv}_{xy}\,,$$

where $[T]_{xy}$ and $[T]_{uv}$ denote the matrix representing T with respect to the x, y and u, v coordinates respectively; $[I]^{xy}_{uv}$ is the matrix that changes from u, v coordinates to x, y coordinates, and $[I]^{uv}_{xy}$ is the matrix that changes coordinates the other way.

The three matrices on the right-hand side are all easily derived from the description we gave for the reflection T:

$$[I]^{xy}_{uv} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}, \quad [T]_{uv} = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \quad [I]^{uv}_{xy} = \left([I]^{xy}_{uv}\right)^{-1} = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}.$$

Computing the matrix product (with the help of the double angle identity) yields:

$$[T]_{xy} = \begin{bmatrix} \cos 2\theta & \sin 2\theta \\ \sin 2\theta & -\cos 2\theta \end{bmatrix}. \tag{1}$$

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For the information of the reader, we note that there are other ways of "deriving" this result. One is by the use of a diagram, which would show that $(1, 0)$ gets reflected to $(\cos 2\theta, \sin 2\theta)$ and $(0, 1)$ gets reflected to $(\sin 2\theta, -\cos 2\theta)$. Another way is to observe that we can rotate an arbitrary mirror line onto the x-axis, then reflect across the x-axis, and rotate back. (The matrix product $[T]_{xy}$ can be seen as operating this way.) We took neither of these two approaches, because to justify them rigorously takes a bit of work, that is avoided by the pure linear algebra approach.

Note also that $[T]_{uv}$ and $[T]_{xy}$ are orthogonal matrices, with determinant -1, as expected.

Reflection across a line of given direction vector

Suppose instead of being given an angle θ, we are given the unit direction vector u to reflect the vector w. We can derive the matrix for the reflection directly, without involving any trigonometric functions.

In the decomposition $\mathbf{w} = a\mathbf{u} + b\mathbf{v}$, we note that $b = \mathbf{w} \cdot \mathbf{v}$. Therefore

$$\mathbf{w}' = (a\mathbf{u} + b\mathbf{v}) - 2b\mathbf{v} = \mathbf{w} - 2(\mathbf{w} \cdot \mathbf{v})\mathbf{v}\,.$$

(In fact, this is the formula used in the source code to draw the diagram in this entry.) To derive the matrix with respect to x, y coordinates, we resort to a trick:

$$\mathbf{w}' = I\mathbf{w} - 2\mathbf{v}(\mathbf{w} \cdot \mathbf{v}) = I\mathbf{w} - 2\mathbf{v}(\mathbf{v}^{\mathrm{tr}}\mathbf{w}) = I\mathbf{w} - 2(\mathbf{v}\mathbf{v}^{\mathrm{tr}})\mathbf{w}\,.$$

Therefore the matrix of the transformation is

$$I - 2\mathbf{v}\mathbf{v}^{\mathrm{tr}} = \begin{bmatrix} u_x^2 - u_y^2 & 2u_x u_y \\ 2u_x u_y & u_y^2 - u_x^2 \end{bmatrix}, \quad \mathbf{u} = (u_x, u_y)^{\mathrm{tr}}, \quad \mathbf{v} = (-u_y, u_x)^{\mathrm{tr}}\,.$$

If u was not a unit vector to begin with, it of course suffices to divide by its magnitude before proceeding. Taking this into account, we obtain the following matrix for a reflection about a line with direction $\mathbf{u}$:

$$\frac{1}{u_x^2 + u_y^2} \begin{bmatrix} u_x^2 - u_y^2 & 2u_x u_y \\ 2u_x u_y & u_y^2 - u_x^2 \end{bmatrix}. \tag{2}$$

Notice that if we put $u_x = \cos\theta$ and $u_y = \sin\theta$ in matrix (**??**), we get matrix (**??**), as it should be.

Reflection across a line of given slope

There is another form for the matrix (**??**). We set $m = \tan\theta$ to be the slope of the line of reflection and use the identities:

$$\cos^2\theta = \frac{1}{\tan^2\theta + 1} = \frac{1}{m^2 + 1}$$
$$\cos 2\theta = 2\cos^2\theta - 1$$
$$\sin 2\theta = 2\sin\theta\cos\theta = 2\tan\theta\cos^2\theta = 2m\cos^2\theta\,.$$

When these equations are substituted in matrix (**??**), we obtain an alternate expression for it in terms of m only:

$$\frac{1}{m^2+1}\begin{bmatrix}1-m^2 & 2m \\ 2m & m^2-1\end{bmatrix}. \tag{3}$$

Thus we have derived the matrix for a reflection about a line of slope m.

Alternatively, we could have also substituted $u_x = 1$ and $u_y = m$ in matrix (**??**) to arrive at the same result.

Topology of reflection matrices

Of course, formula (**??**) does not work literally when $m = \pm\infty$ (the line is vertical). However, that case may be derived by taking the limit $|m| \to \infty$ — this limit operation can be justified by considerations of the topology of the space of two-dimensional reflection matrices.

What is this topology? It is the one-dimensional projective plane $\mathbb{RP}^1$, or simply, the "real projective line". It is formed by taking the circle, and identifying opposite points, so that each pair of opposite points specify a unique mirror line of reflection in $\mathbb{R}^2$. Formula (**??**) is a parameterization of $\mathbb{RP}^1$. Note that (**??**) involves the quantity 2θ, not θ, because for a point $(\cos\theta, \sin\theta)$ on the circle, its opposite point $(\cos(\theta+\pi), \sin(\theta+\pi))$ specify the same reflection, so formula (**??**) has to be invariant when θ is replaced by $\theta + \pi$.

But (**??**) might as well be written

$$[T]_{xy} = \begin{bmatrix}\cos\phi & \sin\phi \\ \sin\phi & -\cos\phi\end{bmatrix}. \tag{4}$$

where $\phi = 2\theta$. For this parameterization of RP^1 to be one-to-one, ϕ can range over interval $(0, 2\pi)$, and the endpoints at $\phi = 0, 2\pi$ overlap just as for a circle, without identifying pairs of opposite points. What does this mean? It is the fact that $\mathbb{RP}^1$ is homeomorphic to the circle S^1.

The real projective line $\mathbb{RP}^1$ is also the one-point compactification of $\mathbb{R}$ (i.e. $\mathbb{RP}^1 = \mathbb{R} \cup \{\infty\}$), as shown by formula (**??**); the number $m = \infty$ corresponds to a reflection across the vertical axis. Note that this "∞" is not the same as the usual $\pm\infty$, because here $-\infty$ and ∞ are actually the same number, both representing the slope of a vertical line.