

Bézier curve*

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1 Introduction

Bézier curves are a particular family of curves in \mathbb{R}^m that parameterize a smooth path between given points. They are often used to model or approximate general curved shapes because of their algebraic simplicity.

Bézier curves were originally developed by Paul de Casteljau in 1959. But it became a famous shape only when Pierre Bézier, a French engineer at Renault, used it to design automobiles in the 1970's. They are now widely used in many fields such as industrial and computer-aided design, vector-based drawing, font design, and 3D modeling.

2 Definition

Given $n + 1$ points \mathbf{P}_i in \mathbb{R}^m , the *Bézier curve* for these *control points* is the curve $\mathbf{B}: [0, 1] \rightarrow \mathbb{R}^m$ defined by

$$\mathbf{B}(t) = \sum_{i=0}^n \binom{n}{i} (1-t)^{n-i} t^i \mathbf{P}_i.$$

The coefficients in this expression:

$$b_{i,n}(t) = \binom{n}{i} (1-t)^{n-i} t^i, \quad i = 0, \dots, n.$$

are the Bernstein basis polynomials.

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3 Special cases

A Bézier curve between two control points is simply the line segment between the two points:

$$\mathbf{B}(t) = (1 - t) \mathbf{P}_0 + t \mathbf{P}_1 .$$

For three control points, we have the quadratic Bézier curve:

$$\mathbf{B}(t) = (1 - t)^2 \mathbf{P}_0 + 2t(1 - t) \mathbf{P}_1 + t^2 \mathbf{P}_2 .$$

For four control points, we have the cubic Bézier curve:

$$\mathbf{B}(t) = (1 - t)^3 \mathbf{P}_0 + 3t(1 - t)^2 \mathbf{P}_1 + 3t^2(1 - t) \mathbf{P}_2 + t^3 \mathbf{P}_3 .$$

These are the most commonly employed versions of Bézier curves.

4 Properties

- Except for the first and the last points, where the curve starts and ends, the Bézier curve should not pass through any of the control points. i.e. all the controls points lie on the curve if and only if the curve is a straight line.
- The curve is tangent to the segment of line $\overline{\mathbf{P}_0 \mathbf{P}_1}$ at the start and $\overline{\mathbf{P}_{n-1} \mathbf{P}_n}$ at the end.
- It is easy to keep until the first order continuity when chaining curves. Moreover, every Bézier curve can be cut at any point into two new Bézier curves. In fact, the design industry uses series of Bézier curves with only 4 control points (a bicubic version of the equations) to draw complex shapes.
- The curves never do fancy stuff and always remain within the convex hull of the control points.
- For one and two points the definition is still valid and the result became a point and a segment of line, respectively.
- But it is impossible to draw circles or arcs of circle.
- It is also impossible to draw two parallel curves (except in trivial cases: straight line Béziers and coincident Béziers curves)
- If the first and the last point are the same, the curve is closed. And if \mathbf{P}_1 and \mathbf{P}_{n-1} are aligned with \mathbf{P}_0 then the curve is closed with a first order continuity.

5 Curve Cut

Section in progress... started the 21th of september 2009... if it seems abandoned while unfinished, delete this section

Let $\mathbf{B}_p(t)$ be a Bezier curve, defined by its control points \mathbf{P}_0 to \mathbf{P}_n for $t \in [0, 1]$.

Let choose a, b such as $[a, b] \subset [0, 1]$ Let take $u = \frac{t-a}{b-a}$ such as $\forall u \in [0, 1], t \in [a, b]$

$\mathbf{B}_q(u)$ is a part of the initial bezier curve with new control points \mathbf{Q}_0 to \mathbf{Q}_n whose coordinates will be determined.

For n control points $\mathbf{B}(t)$ is a $n - 1$ -order polynom.

$$\mathbf{B}_p(a) = \mathbf{B}_q(0)$$

$$\mathbf{B}_p(b) = \mathbf{B}_q(1)$$

Let take $n - 2$ more points, evenly spaced in $[a, b]$ $u_i = k/n - 1 \forall k \in [1..n-1]$

$$\mathbf{B}_p(u(b-a) + a) = \mathbf{B}_q(u_i) \text{ for each } u_i$$

For a 4 points bezier curve Let's take u in $(0, \frac{1}{3}, \frac{2}{3}, 1)$

ie t in $(a, \frac{b-2a}{3}, \frac{2b-a}{3}, b)$

$$\mathbf{Q}_0 = \mathbf{B}_p(a)$$

$$\frac{8}{27}\mathbf{Q}_0 + \frac{4}{9}\mathbf{Q}_1 + \frac{2}{9}\mathbf{Q}_2 + \frac{1}{27}\mathbf{Q}_3 = \mathbf{B}_p(a)$$

$$\frac{1}{27}\mathbf{Q}_0 + \frac{2}{9}\mathbf{Q}_1 + \frac{4}{9}\mathbf{Q}_2 + \frac{8}{27}\mathbf{Q}_3 = \mathbf{B}_p(a)$$

$$\mathbf{Q}_3 = \mathbf{B}_p(b)$$

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