finite difference*

CWoo†

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Definition of Δ.

The derivative of a function $f\colon\mathbb{R}\to\mathbb{R}$ is defined to be the expression

$$\frac{df}{dx}:=\lim_{h\to 0}\frac{f(x+h)-f(x)}{h},$$

which makes sense whenever f is differentiable (at least at x). However, the expression

$$\frac{f(x+h)-f(x)}{h}$$

makes sense even without f being continuous, as long as $h\neq 0$. The expression is called a *finite difference*. The simplest case when $h=1$, written

$$\Delta f(x):=f(x+1)-f(x),$$

is called the *forward difference* of f. For other non-zero h, we write

$$\Delta_h f(x):=\frac{f(x+h)-f(x)}{h}.$$

When $h=-1$, it is called a *backward difference* of f, sometimes written $\nabla f(x):=\Delta_{-1}f(x)$. Given a function $f(x)$ and a real number $h\neq 0$, if we define $y=\frac{x}{h}$ and $g(y)=\frac{f(hy)}{h}$, then we have

$$\Delta g(y)=\Delta_h f(x).$$

Conversely, given $g(y)$ and $h\neq 0$, we can find $f(x)$ such that $\Delta g(y)=\Delta_h f(x)$.

Some Properties of Δ.

It is easy to see that the forward difference operator Δ is linear:

1. $\Delta(f+g)=\Delta(f)+\Delta(g)$
2. $\Delta(cf)=c\Delta(f)$, where $c\in\mathbb{R}$ is a constant.

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Δ also has the properties

1. $\Delta(c) = 0$ for any real-valued constant function c, and
2. $\Delta(I) = 1$ for the identity function $I(x) = x$. constant.

The behavior of Δ in this respect is similar to that of the derivative operator. However, because the continuity of f is not assumed, $\Delta f = 0$ does not imply that f is a constant. f is merely a periodic function $f(x+1) = f(x)$. Other interesting properties include

1. $\Delta a^x = (a-1)a^x$ for any real number a
2. $\Delta x^{(n)} = nx^{(n-1)}$ where $x^{(n)}$ denotes the falling factorial polynomial
3. $\Delta b_n(x) = nx^{n-1}$, where $b_n(x)$ is the Bernoulli polynomial of order n.

From Δ, we can also form other operators. For example, we can iteratively define

$$\Delta^1 f := \Delta f \tag{1}$$
$$\Delta^k f := \Delta(\Delta^{k-1} f), \quad \text{where } k > 1. \tag{2}$$

Of course, all of the above can be readily generalized to Δ_h. It is possible to show that $\Delta_h f$ can be written as a linear combination of

$$\Delta f, \Delta^2 f, \dots, \Delta^h f.$$

Difference Equation.

Suppose $F\colon \mathbb{R}^n \to \mathbb{R}$ is a real-valued function whose domain is the n-dimensional Euclidean space. A *difference equation* (in one variable x) is the equation of the form

$$F(x, \Delta_{h_1}^{k_1} f, \Delta_{h_2}^{k_2} f, \dots, \Delta_{h_n}^{k_n} f) = 0,$$

where $f := f(x)$ is a one-dimensional real-valued function of x. When h_i are all integers, the expression on the left hand side of the difference equation can be re-written and simplified as

$$G(x, f, \Delta f, \Delta^2 f, \dots, \Delta^m f) = 0.$$

Difference equations are used in many problems in the real world, one example being in the study of traffic flow.