Let $A$ be a square complex matrix, $R_i = \sum_{j \neq i} |a_{ij}| \quad 1 \leq i \leq n$. Let’s consider the ovals of this kind: $O_{ij} = \{ z \in \mathbb{C} : |z - a_{ii}| \leq R_i R_j \}$ $\forall i \neq j$. Such ovals are called Cassini ovals.

Theorem (A. Brauer): All the eigenvalues of $A$ lie inside the union of these $n(n-1)/2$ ovals of Cassini:

$$\sigma(A) \subseteq \bigcup_{i \neq j} O_{ij}.$$

Proof: Let $(\lambda, v)$ be an eigenvalue-eigenvector pair for $A$, and let $v_p, v_q$ be the components of $v$ with the two maximal absolute values, that is $|v_p| \geq |v_q| \geq |v_i| \quad \forall i \neq p$. (Note that $|v_p| \neq 0$, otherwise $v$ should be all-zero, in contrast with eigenvector definition). We can also assume that $|v_q|$ is not zero, because otherwise $Av = \lambda v$ would imply $a_{pp} = \lambda$, which trivially verifies the thesis.

Then, since $Av = \lambda v$, we have:

$$(\lambda - a_{pp})v_p = \sum_{j=1, j \neq p}^n a_{pj}v_j$$

and so

$$|\lambda - a_{pp}| |v_p| = \left| \sum_{j=1, j \neq p}^n a_{pj}v_j \right| \leq \sum_{j=1, j \neq p}^n |a_{pj}| |v_j| \leq \sum_{j=1, j \neq p}^n |a_{pj}| |v_q| = R_p |v_q|$$

that is

$$|\lambda - a_{pp}| \leq R_p |v_q|.$$ 

In the same way, we obtain:

$$|\lambda - a_{qq}| \leq R_q |v_p|.$$ 

Multiplying the two inequalities, the two fractional terms vanish, and we get:

$$|\lambda - a_{pp}| |\lambda - a_{qq}| \leq R_p R_q$$

which is the thesis. $\square$

Remarks:

1) Much like the Levy-Desplanques theorem states a sufficient condition, based on Gerschgorin circles, for non-singularity of a matrix, Brauer’s theorem can be employed to state a similar sufficient condition; namely, the following result of Ostrowski holds:

Corollary: Let $A$ be a $n \times n$ complex-valued matrix; if for all $i \neq j$ we have $|a_{ii}| > R_i R_j$, then $A$ is non singular.
The proof is obvious, since, by Brauer’s theorem, the above condition excludes the point \( z = 0 \) from the spectrum of \( A \), implying this way \( \det(A) \neq 0 \).

2) Since both Gerschgorin’s and Brauer’s results rely upon the same \( 2n \) numbers, namely \( \{a_{ii}\}_{i=1}^n \) and \( \{R_i\}_{i=1}^n \), one may wonder if Brauer’s result is stronger than Gerschgorin’s one; actually, the answer is positive, as the following inclusion shows:

**Corollary:** Let \( G(A) = \bigcup_{i=1}^n D_i(A) \) and \( B(A) = \bigcup_{i \neq j}^n O_{ij}(A) \) be respectively Gershgorin and Brauer eigenvalues inclusion regions (\( D_i(A) \) are the Gerschgorin circles and \( O_{ij}(A) \) are the Brauer’s Cassini ovals); then

\[
B(A) \subseteq G(A).
\]

Proof: Let \( O_{ij} \) be one of the \( n(n-1)/2 \) ovals of Cassini for matrix \( A \) and be \( z \in O_{ij} \). If \( R_i = 0 \) or \( R_j = 0 \), Brauer’s theorem imply \( z = a_{ii} \) or \( z = a_{jj} \) respectively; but since both \( a_{ii} \) and \( a_{jj} \) belong to their respective Gerschgorin circles, we have \( z \in (D_i \cup D_j) \). If both \( R_i > 0 \) and \( R_j > 0 \), then we can write:

\[
\frac{|z-a_{ii}|}{R_i} \cdot \frac{|z-a_{jj}|}{R_j} \leq 1.
\]

For the left-hand side to be not greater than 1, \( \frac{|z-a_{ii}|}{R_i} \) or \( \frac{|z-a_{jj}|}{R_j} \) must be not greater than 1, which in turn means \( z \in D_i \) or \( z \in D_j \), that is \( z \in (D_i \cup D_j) \).

3) It’s obvious from definition that there are infinitely many matrices which generate the same ovals of Cassini: namely, let’s define

\[
\Omega(A) = \{ M \in \mathbb{C}^{n \times n} : m_{ii} = a_{ii}, R_i(M) = R_i(A) \}
\]

as the set of all matrices which share the same ovals of Cassini as \( A \). Then, by Brauer’s theorem, we have, for all \( M \in \Omega \) matrices,

\[
\sigma(M) \subseteq B(A),
\]

and therefore, having defined \( \sigma(\Omega) = \bigcup_{M \in \Omega} \sigma(M) \), we have

\[
\sigma(\Omega) \subseteq B(A).
\]

One may then ask how sharp this inclusion is, which, informally speaking, is equivalent to asking how ”efficient” is the ”use”, by Brauer’s theorem, of the \( 2n \) pieces of information \( \{a_{ii}\}_{i=1}^n \) and \( \{R_i\}_{i=1}^n \) in the construction of inclusion sets (if for example we found the inclusion to be very loose, that is \( \sigma(\Omega) \) to be a very little subset of \( B(A) \), we could conjecture that the knowledge of the \( 2n \) numbers used by Brauer’s theorem should have led to a more precise bounding, since the spectra of all matrices which share these numbers lie in a much smaller region). It has been proven that actually

\[
\sigma(\Omega) = B(A),
\]

thus showing Brauer’s ovals are optimal ones under this point of view.

**References**

