units of real cubic fields with exactly one real embedding*

Wkbj79[†]

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Let $K \subseteq \mathbb{R}$ be a number field with $[K:\mathbb{Q}] = 3$ such that K has exactly one real embedding. Thus, $r = 1$ and $s = 1$. Let ${\mathcal{O}_K}^*$ denote the group of units of the ring of integers of K. By Dirichlet's unit theorem, ${\mathcal{O}_K}^* \cong \mu(K) \times \mathbb{Z}$ since $r+s-1 = 1$. The only roots of unity in K are 1 and -1 because $K \subseteq \mathbb{R}$. Thus, $\mu(K) = \{1, -1\}$. Therefore, there exists $u \in {\mathcal{O}_K}^*$ with $u > 1$, such that every element of ${\mathcal{O}_K}^*$ is of the form $\pm u^n$ for some $n \in \mathbb{Z}$.

Let $\rho > 0$ and $0 < \theta < \pi$ such that the conjugates of u are $\rho e^{i\theta}$ and $\rho e^{-i\theta}$. Since u is a unit, $N(u) = \pm 1$. Thus, $\pm 1 = N(u) = u(\rho e^{i\theta})(\rho e^{-i\theta}) = u\rho^2$. Since $u > 0$ and $\rho^2 > 0$, it must be the case that $u\rho^2 = 1$. Thus, $u = \dfrac{1}{\rho^2}$. One can then deduce that $\operatorname{disc} u = -4\sin^2\theta\left(\rho^3 + \dfrac{1}{\rho^3} - 2\cos\theta\right)^2$. Since the maximum value of the polynomial $4\sin^2\theta(x - 2\cos\theta)^2 - 4x^2$ is at most 16, one can deduce that $|\operatorname{disc} u| \le 4\left(u^3 + \dfrac{1}{u^3} + 4\right)$. Define $d = |\operatorname{disc} \mathcal{O}_K|$. Then $d \le |\operatorname{disc} u| \le 4\left(u^3 + \dfrac{1}{u^3} + 4\right)$. Thus, $u^3 \ge \dfrac{d}{4} - 4 - \dfrac{1}{u^3}$. From this, one can obtain that $u^3 \ge \dfrac{d - 16 + \sqrt{d^2 - 32d + 192}}{8}$. (Note that a higher lower bound on u^3 is desirable, and the one stated here is much higher than that stated in Marcus.) Thus, $u^2 \ge \left(\dfrac{d - 16 + \sqrt{d^2 - 32d + 192}}{8}\right)^{\frac{2}{3}}$. Therefore, if an element $x \in {\mathcal{O}_K}^*$ can be found such that $1 < x < \left(\dfrac{d - 16 + \sqrt{d^2 - 32d + 192}}{8}\right)^{\frac{2}{3}}$, then $x = u$.

Following are some applications:

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- The above is most applicable for finding the fundamental unit of a ring of integers of a pure cubic field. For example, if $K = \mathbb{Q}(\sqrt[3]{2})$, then $d = 108$, and the lower bound on u^2 is $\left(\frac{23 + 10\sqrt{21}}{2}\right)^{\frac{2}{3}}$, which is larger than 9. Note that $\left(\sqrt[3]{4} + \sqrt[3]{2} + 1\right)\left(\sqrt[3]{2} - 1\right) = 2-1 = 1$. Since $1 < \sqrt[3]{4}+\sqrt[3]{2}+1 < 9$, it follows that $\sqrt[3]{4} + \sqrt[3]{2} + 1$ is the fundamental unit of $\mathcal{O}_K$.

- The above can also be used for any number field K with $[K:\mathbb{Q}] = 3$ such that K has exactly one real embedding. Let σ be the real embedding. Then the above produces the fundamental unit u of $\sigma(K)$. Thus, $\sigma^{-1}(u)$ is a fundamental unit of K.

References

[1] Marcus, Daniel A. *Number Fields*. New York: Springer-Verlag, 1977.