value of the Riemann zeta function at $s=0$*

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Theorem. *Let ζ denote the meromorphic extension of the Riemann zeta function to the entire complex plane. Then $\zeta(0)=\dfrac{-1}{2}$.*

Proof. Recall that one of the formulas for the Riemann zeta function in the critical strip is given by

$$\zeta(s)=\frac{1}{s-1}+1-s\int_{1}^{\infty}\frac{x-[x]}{x^{s+1}}\,dx,$$

where $[x]$ denotes the integer part of x.

Also recall the functional equation

$$\zeta(s)=2^{s}\pi^{s-1}\sin\frac{\pi s}{2}\Gamma(1-s)\zeta(1-s),$$

where Γ denotes the gamma function.

The only pole of ζ occurs at $s=1$. Therefore, ζ is analytic, and thus continuous, at $s=0$.

Let $\lim\limits_{s\to 0^{+}}$ denote the limit as s approaches 0 along any path contained in the region $\operatorname{Re}(s)>0$. Thus:

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$$
\begin{aligned}
\zeta(0) &= \lim_{s\to 0^+} \zeta(s) \\
&= \lim_{s\to 0^+} 2^s \pi^{s-1} \sin \frac{\pi s}{2} \Gamma(1-s)\zeta(1-s) \\
&= \lim_{s\to 0^+} 2^s \pi^{s-1} \left(\sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} \left(\frac{\pi s}{2}\right)^{2n+1} \right) \Gamma(1-s) \left(\frac{1}{(1-s)-1} + 1 - (1-s) \int_1^\infty \frac{x-[x]}{x^{(1-s)+1}}\, dx \right) \\
&= \lim_{s\to 0^+} 2^s \pi^{s-1} \left(\frac{\pi s}{2}\right) \left(\sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} \left(\frac{\pi s}{2}\right)^{2n} \right) \Gamma(1-s) \left(\frac{1}{-s} + 1 - (1-s) \int_1^\infty \frac{x-[x]}{x^{2-s}}\, dx \right) \\
&= \lim_{s\to 0^+} 2^s \pi^{s-1} \left(\frac{\pi}{2}\right) \left(1 + \sum_{n=1}^{\infty} \frac{(-1)^n}{(2n+1)!} \left(\frac{\pi s}{2}\right)^{2n} \right) \Gamma(1-s) s \left(\frac{-1}{s} + 1 - (1-s) \int_1^\infty \frac{x-[x]}{x^{2-s}}\, dx \right) \\
&= \lim_{s\to 0^+} 2^{s-1} \pi^{s} \left(1 + \sum_{n=1}^{\infty} \frac{(-1)^n}{(2n+1)!} \left(\frac{\pi s}{2}\right)^{2n} \right) \Gamma(1-s) \left(-1 + s - s(1-s) \int_1^\infty \frac{x-[x]}{x^{2-s}}\, dx \right) \\
&= \left(\lim_{s\to 0^+} 2^{s-1} \pi^{s} \Gamma(1-s) \left(-1 + s - s(1-s) \int_1^\infty \frac{x-[x]}{x^{2-s}}\, dx \right) \right) \left(\lim_{s\to 0^+} 1 + \sum_{n=1}^{\infty} \frac{(-1)^n}{(2n+1)!} \left(\frac{\pi s}{2}\right)^{2n} \right) \\
&= \left(2^{0-1} \pi^{0} \Gamma(1-0) \left(-1 + 0 - 0(1-0) \int_1^\infty \frac{x-[x]}{x^{2-0}}\, dx \right) \right) \left(1 + \sum_{n=1}^{\infty} \frac{(-1)^n}{(2n+1)!} \left(\frac{\pi \cdot 0}{2}\right)^{2n} \right) \\
&= \left(\frac{1}{2} \cdot 1 \cdot \Gamma(1) \cdot (-1 + 0 - 0) \right) \left(1 + \sum_{n=1}^{\infty} 0 \right) \\
&= \frac{-1}{2}.
\end{aligned}
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$\square$