Fréchet derivative is unique*

Mathprof[†]

2013-03-21 20:58:57

Theorem The Fréchet derivative is unique.
Proof. Assume that both A and B in $L(\mathsf{V},\mathsf{W})$ satisfy the condition for the Fréchet derivative at the point $\mathbf{x}$. To prove that they are equal we will show that for all $\varepsilon > 0$ the operator norm $\|A-B\|$ is not greater than ε. By the definition of limit there exists a positive δ such that for all $\|\mathbf{h}\| \le \delta$

$$\|f(\mathbf{x}+\mathbf{h}) - f(\mathbf{x}) - A\mathbf{h}\| \le \frac{\varepsilon}{2} \cdot \|\mathbf{h}\| \text{ and } \|f(\mathbf{x}+\mathbf{h}) - f(\mathbf{x}) - B\mathbf{h}\| \le \frac{\varepsilon}{2} \cdot \|\mathbf{h}\|$$

holds. This gives

$$\begin{aligned} \|(A-B)\mathbf{h}\| &= \|(f(\mathbf{x}+\mathbf{h}) - f(\mathbf{x}) - A\mathbf{h}) - (f(\mathbf{x}+\mathbf{h}) - f(\mathbf{x}) - B\mathbf{h})\| \\ &\le \|f(\mathbf{x}+\mathbf{h}) - f(\mathbf{x}) - A\mathbf{h}\| + \|f(\mathbf{x}+\mathbf{h}) - f(\mathbf{x}) - B\mathbf{h}\| \\ &< \varepsilon \cdot \|\mathbf{h}\|. \end{aligned}$$

Now we have

$$\delta \cdot \|A-B\| = \delta \cdot \sup_{\|\mathbf{g}\|\le 1} \|(A-B)\mathbf{g}\| = \sup_{\|\mathbf{g}\|\le \delta} \|(A-B)\mathbf{g}\| \le \sup_{\|\mathbf{g}\|\le \delta} \varepsilon \cdot \|\mathbf{g}\| \le \varepsilon \cdot \delta,$$

thus $\|A-B\| \le \varepsilon$ as we wanted to show.

*⟨*FrechetDerivativeIsUnique*⟩ created: ⟨*2013-03-21*⟩ by: ⟨*Mathprof*⟩ version: ⟨*38221*⟩ Privacy setting: ⟨*1*⟩ ⟨*Theorem*⟩ ⟨*46G05*⟩