

Fréchet derivative is unique*

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Theorem The Fréchet derivative is unique.

Proof. Assume that both A and B in $L(V, W)$ satisfy the condition for the Fréchet derivative at the point \mathbf{x} . To prove that they are equal we will show that for all $\varepsilon > 0$ the operator norm $\|A - B\|$ is not greater than ε . By the definition of limit there exists a positive δ such that for all $\|\mathbf{h}\| \leq \delta$

$$\|f(\mathbf{x} + \mathbf{h}) - f(\mathbf{x}) - A\mathbf{h}\| \leq \frac{\varepsilon}{2} \cdot \|\mathbf{h}\| \text{ and } \|f(\mathbf{x} + \mathbf{h}) - f(\mathbf{x}) - B\mathbf{h}\| \leq \frac{\varepsilon}{2} \cdot \|\mathbf{h}\|$$

holds. This gives

$$\begin{aligned} \|(A - B)\mathbf{h}\| &= \|(f(\mathbf{x} + \mathbf{h}) - f(\mathbf{x}) - A\mathbf{h}) - (f(\mathbf{x} + \mathbf{h}) - f(\mathbf{x}) - B\mathbf{h})\| \\ &\leq \|f(\mathbf{x} + \mathbf{h}) - f(\mathbf{x}) - A\mathbf{h}\| + \|f(\mathbf{x} + \mathbf{h}) - f(\mathbf{x}) - B\mathbf{h}\| \\ &< \varepsilon \cdot \|\mathbf{h}\|. \end{aligned}$$

Now we have

$$\delta \cdot \|A - B\| = \delta \cdot \sup_{\|\mathbf{g}\| \leq 1} \|(A - B)\mathbf{g}\| = \sup_{\|\mathbf{g}\| \leq \delta} \|(A - B)\mathbf{g}\| \leq \sup_{\|\mathbf{g}\| \leq \delta} \varepsilon \cdot \|\mathbf{g}\| \leq \varepsilon \cdot \delta,$$

thus $\|A - B\| \leq \varepsilon$ as we wanted to show.

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