

# semigroup with two elements<sup>\*</sup>

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Perhaps the simplest non-trivial example of a semigroup which is not a group is a particular semigroup with two elements. The underlying set of this semigroup is  $\{a, b\}$  and the operation is defined as follows:

$$\begin{aligned}a \cdot a &= a \\a \cdot b &= b \\b \cdot a &= b \\b \cdot b &= b\end{aligned}$$

It is rather easy to check that this operation is associative, as it should be:

$$\begin{aligned}a \cdot (a \cdot a) &= a \cdot a = a = a \cdot a = (a \cdot a) \cdot a \\a \cdot (a \cdot b) &= a \cdot b = b = a \cdot b = (a \cdot a) \cdot b \\a \cdot (b \cdot b) &= a \cdot b = b = b \cdot b = (a \cdot b) \cdot b \\b \cdot (a \cdot a) &= b \cdot a = b = a \cdot a = (a \cdot a) \cdot a \\b \cdot (a \cdot b) &= b \cdot b = b = b \cdot b = (a \cdot b) \cdot b \\b \cdot (b \cdot a) &= b \cdot b = b = b \cdot a = (b \cdot b) \cdot a \\b \cdot (b \cdot b) &= b \cdot b = b = b \cdot b = (b \cdot b) \cdot b\end{aligned}$$

It is worth noting that this semigroup is commutative and has an identity element, which is  $a$ . It is not a group because the element  $b$  does not have an inverse. In fact, it is not even a cancellative semigroup because we cannot cancel the  $b$  in the equation  $a \cdot b = b \cdot b$ .

This semigroup also arises in various contexts. For instance, if we choose  $a$  to be the truth value "true" and  $b$  to be the truth value "false" and the operation  $\cdot$  to be the logical connective "and", we obtain this semigroup in logic. We may also represent it by matrices like so:

$$a = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \quad b = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$$

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