proof of example of medial quasigroup*

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We shall proceed by first showing that the algebraic systems defined in the parent entry are quasigroups and then showing that the medial property is satisfied.

To show that the system is a quasigroup, we need to check the solubility of equations. Let x and y be two elements of G. Then, by definition of $\cdot$, the equation $x \cdot z = y$ is equivalent to

$$f(x) + g(z) + c = y.$$

This is equivalent to

$$g(z) = y - c - f(x).$$

Since g is an automorphism, there will exist a unique solution z to this equation.

Likewise, the equation $z \cdot x = y$ is equivalent to

$$f(z) + g(x) + c = y$$

which, in turn is equivalent to

$$f(z) = y - c - g(x),$$

so we may also find a unique z such that $z \cdot x = y$. Hence, $(G, \cdot)$ is a quasigroup.

To check the medial property, we use the definition of $\cdot$ to conclude that

$$\begin{aligned} (x \cdot y) \cdot (z \cdot w) &= (f(x) + g(y) + c) \cdot (f(z) + g(w) + c) \\ &= f(f(x) + g(y) + c) + g(f(z) + g(w) + c) + c \end{aligned}$$

Since f and g are automorphisms and the group is commutative, this equals

$$f(f(x)) + f(g(y)) + g(f(z)) + g(g(w)) + f(c) + g(c) + c.$$

Since f and g commute this, in turn, equals

$$f(f(x)) + g(f(y)) + f(g(z)) + g(g(w)) + f(c) + g(c) + c.$$

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Using the commutative and associative laws, we may regroup this expression as follows:

$$(f(f(x)) + f(g(z)) + f(c)) + (g(f(y)) + g(g(w)) + g(c)) + c$$

Because f and g are automorphisms, this equals

$$f(f(x) + g(z) + c) + g(f(y) + g(w) + c) + c$$

By defintion of $\cdot$, this equals

$$f(x \cdot z) + g(y \cdot z) + c,$$

which equals $(x \cdot z) \cdot (y \cdot z)$, so we have

$$(x \cdot y) \cdot (z \cdot w) = (x \cdot z) \cdot (y \cdot z).$$

Thus, the medial property is satisfied, so we have a medial quasigroup.