subobject classifier*

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Motivation

Consider a set A and a subset $B \subseteq A$. B can be thought of as a property of A: there is a function $\chi_B : A \to \{0,1\}$, such that $\chi_B(x) = 1$ iff $x \in B$. This function can be seen to be uniquely determined by the subset B, and conversely. If we denote $P(A)$ the set of all subsets of A (the power set of A), and 2^A the set of all functions from A to $2 := \{0,1\}$, then $P(A) \cong 2^A$.

In fact, we have established a commutative diagram

$$@+ = 4pcB[r]^k[d]_{inc}\{1\}[d]^i A[r]_{\chi_B} 2$$

where inc and i are inclusion functions and k is the unique constant function. Any function $A \to 2$ gives rise to a unique set B making the above diagram commute.

Definition

In category theory, a *subobject classifier* is the generalization of the above example, where A is an object of a certain given category $\mathcal{C}$ and B is a subobject of A, $\{1\}$ is replaced by a terminal object, and 2 is replaced by what is known as a *subobject classifier*, or a *truth object*. If we think of the category **Set**, 2 "classifies" elements of a given set as to whether they belong to a certain subset or not, via a characteristic function. If the value of the function is 1, then the element is in that subset, otherwise it is not.

Formally, let $\mathcal{C}$ be a category with a terminal object 1. A *subobject classifier* is an object Ω in $\mathcal{C}$ such that, for any monomorphism $f : B \to A$, there exists a *unique* morphism χ_B such that

$$@+ = 4pcB[r][d]_f 1[d]^\top A@. >[r]^{\chi_B}\Omega$$

is a pullback diagram. χ_B is called the *characteristic morphism* of f and $\top$ is a *truth morphism*.

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In a category with a terminal object 1, a subobject classifier may or may not exist. If it does, it is unique up to isomorphism. Suppose C has a terminal object 1, has pullbacks, and has a subobject Ω. Then for any object X in $\mathcal{C}$, any morphism $f : X \to \Omega$ gives rise to a unique monomorphism $g : A \to X$ via the pull back of f and $\top$:

$$@+ = 4pcA[r][d]_g 1[d]^\top X[r]^f \Omega$$

Since Ω is a subobject classifier, g determines f uniquely as well. So what we have is a one-to-one correspondence

$$\mathrm{Sub}(X) \cong \hom(X, \Omega)$$

between the subobject functor and hom functor. It can be verified that the bijection is actually a natural isomorphism, so that Sub is a representable functor. Conversely, it may be shown that if Sub is representable, then C has a subobject classifier.