periodic functions*

pahio†

2013-03-21 22:21:55

This entry concerns the periodicity of the meromorphic functions.

Theorem 1. If ω is a period of a function f, then also $n\omega$, with n an arbitrary integer, is a period of f.

Proof. For the positive values of n the theorem is easily proved by induction. If n then is any negative integer $-k$, we can write

$$f(z-k\omega) = f((z-k\omega)+k\omega) = f(z)$$

which is true for all z's. Q.E.D.

Note. If a function has no other periods than $\pm\omega, \pm 2\omega, \pm 3\omega, \ldots,$ the function is called *one-periodic* and ω the *prime period* or *primitive period* of the function. Examples of one-periodic functions are the trigonometric functions sine and cosine (with prime period 2π), tangent and cotangent (prime period π), the exponential function and the hyperbolic sine and cosine (with prime period $2i\pi$), hyperbolic tangent and cotangent (prime period $i\pi$).

Theorem 2. The moduli of all periods of a non-constant meromorphic function f have a positive lower bound.

Proof. Antithesis: there are periods of f with arbitrarily little modulus. Thus we could choose a sequence $\omega_1, \omega_2, \ldots$ of the periods such that $\lim_{n\to\infty}\omega_n = 0$. If z_0 is a regularity point of f, we have

$$f(z_0) = f(z_0+\omega_n) \quad \forall\, n = 1,\, 2,\, \ldots,$$

i.e. the function $f(z)-f(z_0)$ has infinitely many zeros $z_0+\omega_n$ $(n = 1, 2, \ldots)$ which have the accumulation point z_0. But then $f(z)-f(z_0)$ vanishes identically (cf. this entry), i.e. $f(z)$ is a constant function. This contradicts the assumption, and therefore the antithesis is wrong. Q.E.D.

Theorem 3. The periods of a non-constant meromorphic function f do not accumulate to a finite point.

Proof. We make the antithesis, that the periods of f have a finite accumulation point z_0. Thus we can choose two periods ω_1 and ω_2 within a disc

*⟨*PeriodicFunctions*⟩ created: ⟨*2013-03-21*⟩ by: ⟨*pahio*⟩ version: ⟨*39106*⟩ Privacy setting: ⟨*1*⟩ ⟨*Topic*⟩ ⟨*30D20*⟩ ⟨*30D05*⟩ ⟨*30A99*⟩

Figure 1: The argument of Theorem 2

Figure 2: The argument of Theorem 3

with center z_0 and with radius an arbitrary positive number ε. The difference $\omega_1-\omega_2$ is also a period. Because $|\omega_1-\omega_2| < 2\varepsilon$, $f(z)$ seems to have periods with arbitrarily little modulus. This contradicts the theorem 2, and so the antithesis is wrong.

The theorems 2 and 3 imply, that the moduli of all periods of the function f have a positive minimum m_1. Let ω_1 be such a period that $|\omega_1| = m_1$. Then each multiple $n\omega_1$ $(n = \pm 1, \pm 2, \ldots)$ is a period. The points of the complex plane corresponding these periods lie all on the same line

$$\arg z = \arg \omega_1 \tag{1}$$

and are situated at regular intervals. The line does not contain points corresponding other periods, since if there were a period ω on the line between the points $\nu\omega_1$ and $(\nu+1)\omega_1$, then the period $\omega-\nu\omega_1$ would have the modulus $< |\omega_1| = m_1$.

Can a function have other periods than those on the line (1)? If there are such ones, then it's rather easy to prove, using the theorem 3, that their distances from this line have a positive minimum m_2. Suppose that ω_2 is such a period giving the minimum distance m_2. Then also all numbers $\omega = n_1\omega_1 + n_2\omega_2$, with $n_1, n_2 \in \mathbb{Z}$, are periods of f. The corresponding points of the complex plane form the vertices of a lattice of congruent parallelograms. Conversely, one can infer that all the periods of f are of the form

$$\omega = n_1\omega_1+n_2\omega_2 \quad (n_1,\, n_2 \in \mathbb{Z}). \tag{2}$$

In fact, if f had some period point other than (2), then one such would be also in the basic parallelogram with the vertices $0, \omega_1, \omega_2, \omega_1+\omega_2$. This however would contradict the minimality of ω_1 and ω_2.

The numbers ω_1 and ω_2 are called the *prime periods* of the function. We have the

Theorem 4. A non-constant meromorphic function has at most two prime periods. Their ratio is not real.

The functions, which have two prime periods, are called *two-periodic*, *doubly periodic* or *elliptic functions.*

Figure 3: The basic period parallelogram

Figure 4: Lattice generated by the prime periods as the basis

References

[1] R. Nevanlinna & V. Paatero: *Funktioteoria*. Kustannusosakeyhtiö Otava. Helsinki (1963).