lecture notes on polynomial interpolation*

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1 Summary of notation and terminology

- multi-evaluation mapping: $\text{ev}_x : \mathcal{P}_m \to \mathbb{R}^n$, $x \in \mathbb{R}^n$. Here $\mathcal{P}_m$ is the vector space of polynomials of degree $m$ or less.
- interpolation mapping: $\text{pol}_x : \mathbb{R}^n \to \mathcal{P}_{n-1}$, $x \in \mathbb{R}^n$ where $x_1, \ldots, x_n$ are distinct;
- Vandermonde matrix and polynomial;
- overdetermined and underdetermined linear system;
- overdetermined and underdetermined interpolation problem

2 The polynomial interpolation problem

Definition 1. Let $x \in \mathbb{R}^n$ be given. Define $\text{ev}_x : \mathcal{P}_m \to \mathbb{R}^n$, the multi-evaluation mapping, to be the linear transformation given by

$$\text{ev}_x : p \mapsto \begin{bmatrix} p(x_1) \\ \vdots \\ p(x_n) \end{bmatrix}, \quad p \in \mathcal{P}_m.$$ 

Problem 2 (Polynomial interpolation). Given $(x_1, y_1), \ldots, (x_n, y_n) \in \mathbb{R}^2$ to find a polynomial $p \in \mathcal{P}_m$ such that $p(x_i) = y_i$, $i = 1, \ldots, n$. Equivalently, given $x, y \in \mathbb{R}^n$ to find the preimage $\text{ev}_x^{-1}(y)$.

Theorem 3. Fix $x \in \mathbb{R}^n$. The multi-evaluation mapping $\text{ev}_x : \mathcal{P}_{n-1} \to \mathbb{R}^n$ is an isomorphism if and only if the components $x_1, \ldots, x_n$ are distinct.

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3 Vandermonde matrix, polynomial, and determinant

Definition 4. For a given $x \in \mathbb{R}^n$, the following matrix

$$VM(x) = VM(x_1, \ldots, x_n) = 
\begin{bmatrix}
1 & x_1^2 & \cdots & x_1^{n-1} \\
1 & x_2^2 & \cdots & x_2^{n-1} \\
\vdots & \vdots & \ddots & \vdots \\
1 & x_n^2 & \cdots & x_n^{n-1}
\end{bmatrix}$$

is called the Vandermonde matrix. The expression,

$$V(x) = V(x_1, \ldots, x_n) = \prod_{1 \leq i < j \leq n} (x_j - x_i)$$

is called the Vandermonde polynomial. Note: we define $V(x_1) = 1$; the usual convention is that the empty product is equal to 1.

Proposition 5. The Vandermonde matrix is the transformation matrix of $ev_x$ with the monomial basis $[1, x, \ldots, x^n]$ as the input basis and the standard basis $[e_1, \ldots, e_{n+1}]$ as the output basis.

Theorem 6. The Vandermonde polynomial gives the determinant of the Vandermonde matrix:

$$\begin{vmatrix}
1 & x_1 & \cdots & x_1^{n-1} \\
1 & x_2 & \cdots & x_2^{n-1} \\
\vdots & \vdots & \ddots & \vdots \\
1 & x_n & \cdots & x_n^{n-1}
\end{vmatrix} = \prod_{1 \leq i < j \leq n+1} (x_j - x_i), \quad (1)$$

or more succinctly,

$$\det VM(x) = V(x).$$

Proof. We will prove formula (1) by induction on $n$. Note that

$$VM(x_1) = [1], \quad V(x_1) = 1.$$

Evidently then, the formula works for $n = 1$. Next, suppose that we believe the formula for a given $n$. We show that the formula is valid for $n + 1$. For $x_1, \ldots, x_n \in \mathbb{R}$ and a variable $x$, and consider the $n^{th}$ degree polynomial

$$p(x) = \det VM(x_1, \ldots, x_n, x) = 
\begin{vmatrix}
1 & x_1 & \cdots & x_1^{n-1} & x_1^n \\
1 & x_2 & \cdots & x_2^{n-1} & x_2^n \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
1 & x_n & \cdots & x_n^{n-1} & x_n^n \\
1 & x & \cdots & x^{n-1} & x^n
\end{vmatrix}$$

or more succinctly,

$$\det VM(x_1, \ldots, x_n, x) = V(x).$$
By the properties of determinants, \(x_1, \ldots, x_n\) are roots of \(p(x)\). Taking the cofactor expansion along the bottom row, we see that the coefficient of \(x^n\) is \(V(x_1, \ldots, x_n)\). Therefore,

\[
p(x) = V(x_1, \ldots, x_n)(x - x_1) \cdots (x - x_n) = V(x_1, \ldots, x_n, x),
\]
as was to be shown.

4 Lagrange interpolation formula

Let \(x_1, \ldots, x_n \in \mathbb{R}\) be distinct. We know that \(ev_x : \mathcal{P}_n \to \mathbb{R}^n\) is invertible. Let \(\text{pol}_x : \mathbb{R}^n \to \mathcal{P}_{n-1}\) denote the inverse. In principle, this inverse is described by the inverse of the Vandermonde matrix. Is there another way to solve the interpolation problem? For \(i = 1, \ldots, n\) let us define the polynomial

\[
p_i(x) = \prod_{j \neq i} \frac{x - x_j}{x_i - x_j}.
\]

These \(n - 1\) degree polynomials have been “engineered” so that \(p_i(x_i) = 1\) and so that \(p_i(x_j) = 0\) for \(i \neq j\)

**Theorem 7** (Lagrange interpolation formula). Let \(x_1, \ldots, x_n \in \mathbb{R}\) be distinct. Then,

\[
\text{pol}_x(y) = y_1p_1 + \cdots + y_np_n.
\]

5 Underdetermined and overdetermined interpolation

**Definition 8.** Let \(T : U \to V\) be a linear transformation of finite dimensional vector spaces. A linear problem is an equation of the form

\[
T(u) = v,
\]

where \(v \in V\) is given, and \(u \in U\) is the unknown. To be more precise, the problem is to determine the preimage \(T^{-1}(v)\) for a given \(v \in V\). We say that the problem is overdetermined if \(\dim V > \dim U\), i.e. if there are more equations than unknowns. The linear problem is said to be underdetermined if \(\dim V < \dim U\), i.e., if there are more variables than equations.

**Remark 9.** By the rank-nullity theorem, an underdetermined linear problem will either be inconsistent, or will have multiple solutions. Thus, an underdetermined system arises when \(T\) is not one-to-one; i.e., the kernel \(\ker(T)\) is non-trivial. An overdetermined linear system is inconsistent, unless \(v\) satisfies a number linear compatibility constraints, equations that describe the image \(\text{Im}(T)\). To put it another way, an overdetermined system arises when \(T\) is not onto.
Definition 10. Let \( x_1, \ldots, x_n \in \mathbb{R} \) be distinct and let \( \text{ev}_x : P_m \to \mathbb{R}^n \) be the corresponding multi-evaluation mapping. Let \( y \in \mathbb{R}^n \) be given. The linear equation
\[
\text{ev}_x(p) = y, \quad p \in P_m
\]
is called an underdetermined interpolation problem if \( m \geq n \). If \( m \leq n - 2 \), we call the above equation an overdetermined interpolation problem. Note that if \( m = n - 1 \), the interpolation problem is “just right”; there is exactly one solution, namely the polynomial \( \text{pol}_x(y) \) given by the Lagrange interpolation formula.

Proposition 11 (Underdetermined interpolation). Let \( x_1, \ldots, x_n \in \mathbb{R} \) be distinct. Define
\[
q(x) = (x - x_1) \cdots (x - x_n)
\]
to be the \( n \)th degree polynomial with the \( x_i \) as its roots. Suppose that \( m \geq n \). Then, the multi-evaluation mapping \( \text{ev}_x : P_m \to \mathbb{R}^n \) is not one-to-one. A basis for the kernel is given by \( q(x), xq(x), \ldots, x^{m-n}q(x) \). Let \( y_1, \ldots, y_n \in \mathbb{R} \) be given. The solution set to the interpolation problem \( \text{ev}_x(p) = y \), is given by
\[
\text{ev}_x^{-1}(y) = \{ r + sq : s \in P_{m-n} \},
\]
where \( r = \text{pol}_x(y) \) is the \( n-1 \)st degree polynomial given by the Lagrange interpolation formula. To put it another way, the general solution of the equation
\[
\text{ev}_x(p) = y
\]
is given by
\[
p = r + sq, \quad s \in P_{n-m} \text{ free}.
\]

Proposition 12 (Overdetermined interpolation). Let \( x_1, \ldots, x_n \in \mathbb{R} \) be distinct. Suppose that \( m \leq n - 2 \). Then, the multi-evaluation mapping \( \text{ev}_x : P_m \to \mathbb{R}^n \) is not onto. If \( m = n - 2 \), then \( y \in \mathbb{R}^4 \) belongs to \( \text{Im}(\text{ev}_x) \) if and only if
\[
\begin{vmatrix}
1 & x_1 & \cdots & x_1^m & y_1 \\
1 & x_2 & \cdots & x_2^m & y_2 \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
1 & x_{n-1} & \cdots & x_{n-1}^m & y_{n-1} \\
1 & x_n & \cdots & x_n^m & y_n
\end{vmatrix} = 0
\]
More generally, for any \( m \leq n - 2 \), the interpolation problem \( \text{ev}_x(p) = y \), \( y \in \mathbb{R}^n \) has a solution if and only if rank \( M(x, y) = m + 1 \) where
\[
M(x, y) =
\begin{bmatrix}
1 & x_1 & \cdots & x_1^m & y_1 \\
1 & x_2 & \cdots & x_2^m & y_2 \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
1 & x_{m+1} & \cdots & x_{m+1}^m & y_{m+1} \\
\vdots & \vdots & \vdots & \vdots & \vdots \\
1 & x_n & \cdots & x_n^m & y_n
\end{bmatrix}
\]
Remark 13. The compatibility constraints on $y_1, \ldots, y_n$ for an overdetermined system amount to the condition that $y$ lie in the image of $ev_x$, or what is equivalent, belong to the column space of the corresponding transformation matrix. We can therefore obtain the constraint equations by row reducing the augmented matrix $M(x, y)$ to echelon form. The back entries of the last $n - m - 1$ rows will hold the constraint equations. Equivalently, we can solve the interpolation problem for $(x_1, y_1), \ldots, (x_m, y_m)$ to obtain a $p = y_1p_1 + y_{m+1}p_{m+1} \in P_m$. The additional equations
\[ y_i = y_1p_1(x_i) + \cdots + y_{m+1}p_{m+1}(x_i), \quad i = m + 2, \ldots, n \]
are the desired compatibility constraints on $y_1, \ldots, y_n$. 