example of a Bezout domain that is not a PID*

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Let $\mathbb{A}$ be the ring of all algebraic numbers whose minimal polynomials are in $\mathbb{Z}[x]$; i.e., every element of $\mathbb{A}$ is an algebraic integer.

In the following example, ideals are considered to be of $\mathbb{A}$ unless indicated otherwise via intersection with a subring of $\mathbb{A}$.

Let I be a finitely generated ideal of $\mathbb{A}$. Then there exists a positive integer n and $\alpha_1, \dots, \alpha_n \in \mathbb{A}$ with $I = \langle \alpha_1, \dots, \alpha_n \rangle$. Let $K = \mathbb{Q}(\alpha_1, \dots, \alpha_n)$, and let $\mathcal{O}_K$ denote the ring of integers of K. Then $\alpha_1, \dots, \alpha_n \in \mathcal{O}_K$ and $I \cap \mathcal{O}_K$ is an ideal of $\mathcal{O}_K$. Let h denote the class number of K. Then $(I \cap \mathcal{O}_K)^h = \langle \beta \rangle \cap \mathcal{O}_K$ for some $\beta \in \mathcal{O}_K$. Let $L = K(\sqrt[h]{\beta})$, and let $\mathcal{O}_L$ denote the ring of integers of L. Then

$$\begin{aligned}
(I \cap \mathcal{O}_L)^h &= [(I \cap \mathcal{O}_K)\mathcal{O}_L]^h \\
&= (I \cap \mathcal{O}_K)^h (\mathcal{O}_L)^h \\
&= (\langle \beta \rangle \cap \mathcal{O}_K)\mathcal{O}_L \\
&= \langle \beta \rangle \cap \mathcal{O}_L \\
&= (\langle \sqrt[h]{\beta} \rangle \cap \mathcal{O}_L)^h
\end{aligned}$$

Since unique factorization of ideals holds in $\mathcal{O}_L$, $I \cap \mathcal{O}_L = \langle \sqrt[h]{\beta} \rangle \cap \mathcal{O}_L$. Since $\mathcal{O}_K \subseteq \mathcal{O}_L$ and $\alpha_1, \dots, \alpha_n \in I \cap \mathcal{O}_K \subseteq I \cap \mathcal{O}_L = \langle \sqrt[h]{\beta} \rangle \cap \mathcal{O}_L$, there exist $\gamma_1, \dots, \gamma_n \in \mathcal{O}_L$ with $\alpha_j = \gamma_j \sqrt[h]{\beta}$ for all positive integers j with $j \leq n$. Thus, $I = \langle \alpha_1, \dots, \alpha_n \rangle = \langle \gamma_1 \sqrt[h]{\beta}, \dots, \gamma_n \sqrt[h]{\beta} \rangle \subseteq \langle \sqrt[h]{\beta} \rangle$. Since $I \subseteq \langle \sqrt[h]{\beta} \rangle$ and $I \cap \mathcal{O}_L = \langle \sqrt[h]{\beta} \rangle \cap \mathcal{O}_L$, $I = \langle \sqrt[h]{\beta} \rangle$. Hence, I is principal. It follows that $\mathbb{A}$ is a Bezout domain.

On the other hand, $\mathbb{A}$ is not a principal ideal domain (PID). For example, the ideal generated by all of the nth roots of 2, $J = \langle 2, \sqrt{2}, \sqrt[3]{2}, \dots \rangle$, is an ideal of $\mathbb{A}$ that is not principal.

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