# curvature determines the curve*

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The curvature of plane curve determines uniquely the form and size of the curve, i.e. one has the

**Theorem.** If $s \mapsto k(s)$ is a continuous real function, then there exists always plane curves satisfying the equation

$$\kappa = k(s) \tag{1}$$

between their curvature $\kappa$ and the arc length $s$. All these curves are congruent.

*Proof.* Suppose that a curve $C$ satisfies the condition (1). Let the value $s = 0$ correspond to the point $P_0$ of this curve. We choose $O$ as the origin of the plane. The tangent and the normal of $C$ in $O$ are chosen as the $x$-axis and the $y$-axis, with positive directions the directions of the tangent and normal vectors of $C$, respectively. According to (1) and the definition of curvature, the equation

$$\frac{d\theta}{ds} = k(s)$$

for the direction angle $\theta$ of the tangent of $C$ is valid in this coordinate system; the initial condition is

$$\theta = 0 \quad \text{when} \quad s = 0.$$

Thus we get

$$\theta = \int_0^s k(t)\,dt =: \vartheta(s), \tag{2}$$

which implies

$$\frac{dx}{ds} = \cos\vartheta(s), \quad \frac{dy}{ds} = \sin\vartheta(s). \tag{3}$$

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Since $x = y = 0$ when $s = 0$, we obtain

$$x \;=\; \int_0^s \cos\vartheta(t)\,dt, \quad y \;=\; \int_0^s \sin\vartheta(t)\,dt. \tag{4}$$

Thus the function $s \mapsto k(s)$ determines uniquely these functions $x$ and $y$ of the parameter $s$, and (4) represents a curve with definite form and size.

The above reasoning shows that every curve which satisfies (1) is congruent with the curve (4).

We have still to show that the curve (4) satisfies the condition (1). By differentiating the equations (4) we get the equations (3), which imply $(\frac{dx}{ds})^2 + (\frac{dy}{ds})^2 = 1$, or $ds^2 = dx^2 + dy^2$ which means that the parameter $s$ represents the arc length of the curve (4), counted from the origin. Differentiating (3) we get, because $\vartheta'(s) = k(s)$ by (2),

$$\frac{d^2x}{ds^2} \;=\; -k(s)\sin\vartheta(s), \quad \frac{d^2y}{ds^2} \;=\; k(s)\cos\vartheta(s). \tag{5}$$

The equations (3) and (5) then yield

$$\frac{dx}{ds}\frac{d^2y}{ds^2} - \frac{dy}{ds}\frac{d^2x}{ds^2} \;=\; k(s),$$

i.e. the curvature of (4), according the parent entry, satisfies

$$\begin{vmatrix} x' & y' \\ x'' & y'' \end{vmatrix} \;=\; k(s).$$

Thus the proof is settled.

# References


[1] Ernst Lindelöf: *Differentiali- ja integralilasku ja sen sovellutukset I.* WSOY. Helsinki (1950).