inversion of plane*

pahio[†]

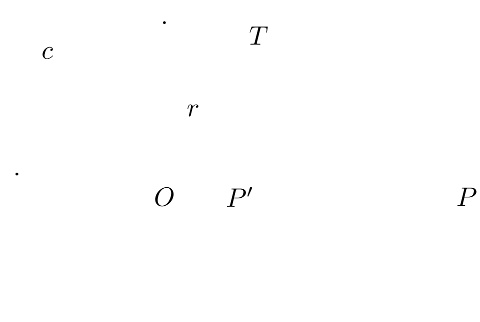
2013-03-21 23:08:44

Let c be a fixed circle in the Euclidean plane with center O and radius r. Set for any point $P \neq O$ of the plane a corresponding point P', called the *inverse point* of P with respect to c, on the closed ray from O through P such that the product

$$P'O \cdot PO$$

has the constant value r^2. This mapping $P \mapsto P'$ of the plane interchanges the inside and outside of the base circle c. The point O' is the "infinitely distant point" of the plane.

The following is an illustration of how to obtain P' for a given circle c and point P outside of c. The restricted tangent from P to c is drawn in blue, the line segment that determines P' (perpendicular to $\overline{OP}$, having an endpoint on $\overline{OP}$, and having its other endpoint at the point of tangency T of the circle and the tangent line) is drawn in red, and the radius $\overline{OT}$ is drawn in green.



The picture justifies the correctness of P', since the triangles $\triangle OPT$ and $\triangle OTP'$ are similar, implying the proportion $PO : TO = TO : P'O$ whence $P'O \cdot PO = (TO)^2 = r^2$. Note that this same argument holds if P and P' were swapped in the picture.

*⟨InversionOfPlane⟩ created: ⟨2013-03-21⟩ by: ⟨pahio⟩ version: ⟨39542⟩ Privacy setting: ⟨1⟩ ⟨Topic⟩ ⟨51K99⟩ ⟨53A30⟩ ⟨30E20⟩

Inversion formulae. If O is chosen as the origin of $\mathbb{R}^2$ and $P = (x, y)$ and $P' = (x', y')$, then

$$x' = \frac{rx}{x^2 + y^2}, \quad y' = \frac{ry}{x^2 + y^2}; \qquad x = \frac{rx'}{x'^{\,2} + y'^{\,2}}, \quad y = \frac{ry'}{x'^{\,2} + y'^{\,2}}.$$

Note. Determining inverse points can also be done in the complex plane. Moreover, the mapping $P \mapsto P'$ is always a Möbius transformation. For example, if $c = \{z \in \mathbb{Z} \colon |z| = 1\}$, i.e. $O = 0$ and $r = 1$, then the mapping $P \mapsto P'$ is given by $f \colon \mathbb{C} \cup \{\infty\} \to \mathbb{C} \cup \{\infty\}$ defined by $f(z) = \dfrac{1}{z}$.

Properties of inversion

- The inversion is *involutory*, i.e. if $P \mapsto P'$, then $P' \mapsto P$.
- The inversion is inversely conformal, i.e. the intersection angle of two curves is preserved (check the Cauchy–Riemann equations!).
- A line through the center O is mapped onto itself.
- Any other line is mapped onto a circle that passes through the center O.
- Any circle through the center O is mapped onto a line; if the circle intersects the base circle c, then the line passes through both intersection points.
- Any other circle is mapped onto its homothetic circle with O as the homothety center.

References

[1] E. J. Nyström: *Korkeamman geometrian alkeet sovellutuksineen.* Kustannusosakeyhtiö Otava, Helsinki (1948).