differential equations for x^x*

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In this entry, we will derive differential equations satisfied by the function x^x. [1] We begin by computing its derivative. To do this, we write $x^x = e^{x \log x}$ and apply the chain rule:

$$\frac{d}{dx} x^x = \frac{d}{dx} e^{x \log x} = e^{x \log x}(1 + \log x) = x^x(1 + \log x)$$

Set $y = x^x$. Then we have $y'/y = 1 + \log x$. Taking another derivative, we have

$$\frac{d}{dx}\left(\frac{y'}{y}\right) = \frac{1}{x}.$$

Applying the quotient rule and simplifying, this becomes

$$yy'' - (y')^2 - y^2/x = 0.$$

It is also possible to derive an equation in which x does not appear. We start by noting that, if $z = 1/x$, then $z' + z^2 = 0$. If, as above, $y = x^x$, we have $(d/dx)(y'/y) = z$. Combining equations,

$$\frac{d^2}{dx^2}\left(\frac{y'}{y}\right) + \left(\frac{d}{dx}\left(\frac{y'}{y}\right)\right)^2 = 0;$$

applying the quotient rule and simplifying,

$$y^3 y''' - y^2 (y'')^2 + 2y(y')^2 y'' - 3y^2 y' y'' - (y')^4 + 2y(y')^3 = 0.$$

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[1]In this entry, we restrict x, and hence x^x to be strictly positive real numbers, hence it is justified to divide by these quantities.