# examples of lamellar field*

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In the examples that follow, show that the given vector field $\vec{U}$ is lamellar everywhere in $\mathbb{R}^3$ and determine its scalar potential $u$.

**Example 1.** Given

$$\vec{U} \;:=\; y\,\vec{i}+(x+\sin z)\,\vec{j}+y\cos z\,\vec{k}.$$

For the rotor (curl) of the field we obtain $\nabla\times\vec{U} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ y & x{+}\sin z & y\cos z \end{vmatrix}$

$= \left(\frac{\partial(y\cos z)}{\partial y}-\frac{\partial(x{+}\sin z)}{\partial z}\right)\vec{i}+\left(\frac{\partial y}{\partial z}-\frac{\partial(y\cos z)}{\partial x}\right)\vec{j}+\left(\frac{\partial(x{+}\sin z)}{\partial x}-\frac{\partial y}{\partial y}\right)\vec{k},$
which is identically $\vec{0}$ for all $x$, $y$, $z$. Thus, by the definition given in the parent entry, $\vec{U}$ is lamellar.
Since $\nabla u = \vec{U}$, the scalar potential $u = u(x,\,y,\,z)$ must satisfy the conditions

$$\frac{\partial u}{\partial x} = y, \quad \frac{\partial u}{\partial y} = x{+}\sin z, \quad \frac{\partial u}{\partial z} = y\cos z.$$

Thus we can write

$$u = \int y\,dx = xy + C_1,$$

where $C_1$ may depend on $y$ or $z$. Differentiating this result with respect to $y$ and comparing to the second condition, we get

$$\frac{\partial u}{\partial y} = x + \frac{\partial C_1}{\partial y} = x + \sin z.$$

Accordingly,

$$C_1 = \int \sin z\,dy = y\sin z + C_2,$$

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where $C_2$ may depend on $z$. So

$$u = xy + y\sin z + C_2.$$

Differentiating this result with respect to $z$ and comparing to the third condition yields

$$\frac{\partial u}{\partial z} = y\cos z + \frac{\partial C_2}{\partial z} = y\cos z.$$

This means that $C_2$ is an arbitrary constant. Thus the form

$$u = xy + y\sin z + C$$

expresses the required potential function.

**Example 2.** This is a particular case in $\mathbb{R}^2$:

$$\vec{U}(x,\, y,\, 0) := \omega y\,\vec{i} + \omega x\,\vec{j}, \quad \omega = \text{constant}$$

Now, $\nabla\times\vec{U} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \omega y & \omega x & 0 \end{vmatrix} = \left(\frac{\partial(\omega x)}{\partial x} - \frac{\partial(\omega y)}{\partial y}\right)\vec{k} = \vec{0}$, and so $\vec{U}$ is lamellar.

Therefore there exists a potential field $u$ with $\vec{U} = \nabla u$. We deduce successively:

$$\frac{\partial u}{\partial x} = \omega y;\;\; u(x,y,0) = \omega xy + f(y);\;\; \frac{\partial u}{\partial y} = \omega x + f'(y) \equiv \omega x;\;\; f'(y) = 0;\;\; f(y) = C$$

Thus we get the result

$$u(x,\, y,\, 0) = \omega xy + C,$$

which corresponds to a particular case in $\mathbb{R}^2$.

**Example 3.** Given

$$\vec{U} := ax\vec{i} + by\vec{j} - (a+b)z\vec{k}.$$

The rotor is now $\nabla\times\vec{U} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ ax & by & -(a+b)z \end{vmatrix} = \vec{0}$. From $\nabla u = \vec{U}$ we obtain

$$\frac{\partial u}{\partial x} = ax \implies u = \frac{ax^2}{2} + f(y,z) \quad (1)$$

$$\frac{\partial u}{\partial y} = by \implies u = \frac{by^2}{2} + g(z,x) \quad (2)$$

$$\frac{\partial u}{\partial z} = -(a+b)z \implies u = -(a+b)\frac{z^2}{2} + h(x,y) \quad (3)$$

Differentiating (1) and (2) with respect to $z$ and using (3) give

$$-(a+b)z = \frac{\partial f(y,z)}{\partial z} \quad \Longrightarrow \quad f(y,z) = -(a+b)\frac{z^2}{2} + F(y) \quad (1');$$

$$-(a+b)z = \frac{\partial g(z,x)}{\partial z} \quad \Longrightarrow \quad g(z,x) = -(a+b)\frac{z^2}{2} + G(x) \quad (2').$$

We substitute $(1')$ and $(2')$ again into (1) and (2) and deduce as follows:

$$u = \frac{ax^2}{2} - (a+b)\frac{z^2}{2} + F(y); \ \ \frac{\partial u}{\partial y} = F'(y) = by; \ \ F(y) = \frac{by^2}{2} + C_1; \ \ f(y,z) = \frac{by^2}{2} - (a+b)\frac{z^2}{2} + C_1 \quad (1'');$$

$$u = \frac{by^2}{2} - (a+b)\frac{z^2}{2} + G(x); \ \ \frac{\partial u}{\partial x} = G'(x) = ax; \ \ G(x) = \frac{ax^2}{2} + C_2; \ \ g(z,x) = \frac{ax^2}{2} - (a+b)\frac{z^2}{2} + C_2 \quad (2'');$$

putting $(1'')$, $(2'')$ into (1), (2) then gives us

$$u = \frac{ax^2}{2} + \frac{by^2}{2} - (a+b)\frac{z^2}{2} + C_1, \quad u = \frac{ax^2}{2} + \frac{by^2}{2} - (a+b)\frac{z^2}{2} + C_2,$$

whence, by comparing, $C_1 = C_2 = C$, so that by (3), the expression $h(x,y)$ and $u$ itself have been found, that is,

$$u = \frac{ax^2}{2} + \frac{by^2}{2} - (a+b)\frac{z^2}{2} + C.$$

Unlike Example 1, the last two examples are also solenoidal, i.e. $\nabla \cdot \vec{U} = 0$, which physically may be interpreted as the continuity equation of an incompressible fluid flow.

**Example 4.** An additional example of a lamellar field would be

$$\vec{U} := -\frac{ay}{x^2+y^2}\vec{i} + \frac{ax}{x^2+y^2}\vec{j} + v(z)\vec{k}$$

with a differentiable function $v : \mathbb{R} \to \mathbb{R}$; if $v$ is a constant, then $\vec{U}$ is also solenoidal.