Chinese remainder theorem in terms of divisor theory*

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In a ring with a divisor theory, a congruence $\alpha \equiv \beta \pmod{\mathfrak{a}}$ with respect to a divisor module $\mathfrak{a}$ means that $\mathfrak{a} \mid \alpha - \beta$.

Theorem. Let $\mathcal{O}$ be an integral domain having the divisor theory $\mathcal{O}^* \to \mathfrak{D}$. For arbitrary pairwise coprime divisors $\mathfrak{a}_1, \ldots, \mathfrak{a}_s$ in $\mathfrak{D}$ and for arbitrary elements $\alpha_1, \ldots, \alpha_s$ of the domain $\mathcal{O}$ there exists an element ξ in $\mathcal{O}$ such that

$$\begin{cases} \xi \equiv \alpha_1 \pmod{\mathfrak{a}_1} \\ \cdots \quad \cdots \quad \cdots \\ \xi \equiv \alpha_s \pmod{\mathfrak{a}_s} \end{cases}$$

Proof. Let

$$\mathfrak{b}_i := \prod_{j \neq i} \mathfrak{a}_j \quad (i = 1, \ldots, s).$$

Apparently, the divisors $\mathfrak{b}_1, \ldots, \mathfrak{b}_s$ are mutually coprime, whence there are in the ring $\mathcal{O}$ the elements $\beta_1, \ldots, \beta_s$ divisible by the divisors $\mathfrak{b}_1, \ldots, \mathfrak{b}_s$, respectively, such that

$$\beta_1 + \ldots + \beta_s = 1. \tag{1}$$

For every $i \neq j$, the divisor $\mathfrak{a}_i$ divides $\mathfrak{b}_j$ and therefore also the element β_j. Then the equation (1) implies that $\beta_i \equiv 1 \pmod{\mathfrak{a}_i}$ and thus the element

$$\xi := \alpha_1\beta_1 + \ldots + \alpha_s\beta_s$$

satisfies

$$\xi \equiv \alpha_i\beta_i \equiv \alpha_i \pmod{\mathfrak{a}_i}$$

for each $i = 1, \ldots, s$. Q.E.D.

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References

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