One way to manufacture an automaton out of existing automata is by taking products.

**Products of Two Automata**

Let $A_1 = (S_1, \Sigma_1, \delta_1, I_1, F_1)$ and $A_2 = (S_2, \Sigma_2, \delta_2, I_2, F_2)$ be two automata. We define the product $A$ of $A_1$ and $A_2$, written $A_1 \times A_2$, as the quintuple 

$$(S, \Sigma, \delta, I, F) := (S_1 \times S_2, \Sigma_1 \times \Sigma_2, \delta_1 \times \delta_2, I_1 \times I_2, F_1 \times F_2),$$

where $\delta$ is a function from $S \times \Sigma$ to $P(S_1) \times P(S_2) \subseteq P(S)$, given by 

$$\delta((s_1, s_2), (\alpha_1, \alpha_2)) := \delta_1(s_1, \alpha_1) \times \delta_2(s_2, \alpha_2).$$

Since $S, \Sigma, I, F$ are non-empty, $A$ is an automaton. The automaton $A$ can be thought of as a machine that runs automata $A_1$ and $A_2$ simultaneously. A pair $(\alpha_1, \alpha_2)$ of symbols being fed into $A$ at start state $(q_1, q_2) \in I$ is the same as $A_1$ reading $\alpha_1$ at state $q_1$ and $A_2$ reading $\alpha_2$ at state $q_2$. The set of all possible next states for the configuration $((s_1, s_2), (\alpha_1, \alpha_2))$ in $A$ is the same as the set of all possible combinations $(t_1, t_2)$, where $t_1$ is a next state for the configuration $(s_1, \alpha_1)$ in $A_1$ and $t_2$ is a next state for the configuration $(s_2, \alpha_2)$ in $A_2$.

If $A_1$ and $A_2$ are FSA, so is $A$. In addition, if both $A_1$ and $A_2$ are deterministic, so is $A$, because 

$$\delta((s_1, s_2), (\alpha_1, \alpha_2)) = (\delta_1(s_1, \alpha_1), \delta_2(s_2, \alpha_2)),$$

and $I$ is a singleton.

As usual, $\delta$ can be extended to read words over $\Sigma$, and it is easy to see that 

$$\delta((s_1, s_2), (a_1, a_2)) = \delta_1(s_1, a_1) \times \delta_2(s_2, a_2),$$

where $a_1$ and $a_2$ are words over $\Sigma_1$ and $\Sigma_2$ respectively. A word $(a_1, a_2)$ is accepted by $A$ iff $a_1$ is accepted by $A_1$ and $a_2$ is accepted by $A_2$. 

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Intersection of Two Automata

Again, we assume $A_1$ and $A_2$ are automata specified above. Now, suppose $\Sigma_1 = \Sigma_2 = \Delta$. Then $\Delta$ can be identified as the diagonal in $\Sigma = \Sigma_1 \times \Sigma_2 = \Delta^2$.

We are then led to an automaton

$$A_1 \cap A_2 := (S, \Delta, \delta, I, F),$$

where $S, I,$ and $F$ are defined previously, and $\delta$ is given by

$$\delta((s_1, s_2), \alpha) = \delta_1(s_1, \alpha) \times \delta_2(s_2, \alpha).$$

Suppose in addition that $\Delta$ is finite. From the discussion in the previous section, it is evident that the language accepted by $A_1 \cap A_2$ is the same as the intersection of the language accepted by $A_1$ and the language accepted by $A_2$:

$$L(A_1 \cap A_2) = L(A_1) \cap L(A_2).$$