a characterization of the radical of an ideal*

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Proposition 1. *Let I be an ideal in a ring R, and $\sqrt{I}$ be its radical. Then $\sqrt{I}$ is the intersection of all prime ideals containing I.*

Proof. Suppose $x \in \sqrt{I}$, and P is a prime ideal containing I. Then $R-P$ is an m-system. If $x \in R-P$, then $(R-P)\cap I \neq \varnothing$, contradicting the assumption that $I \subseteq P$. Therefore $x \notin R-P$. In other words, $x \in P$, and we have one of the inclusions.

Conversely, suppose $x \notin \sqrt{I}$. Then there is an m-system S containing x such that $S \cap I = \varnothing$. Enlarge I to a prime ideal P disjoint from S, so that $x \notin P$ (we can do this; for a proof, see the second remark in this entry). By contrapositivity, we have the other inclusion. □

Remark. This shows that every prime ideal is a radical ideal: for $\sqrt{P}$ is the intersection of all prime ideals containing P, and if P is itself prime, then $P=\sqrt{P}$.

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