Fresnel formulas*

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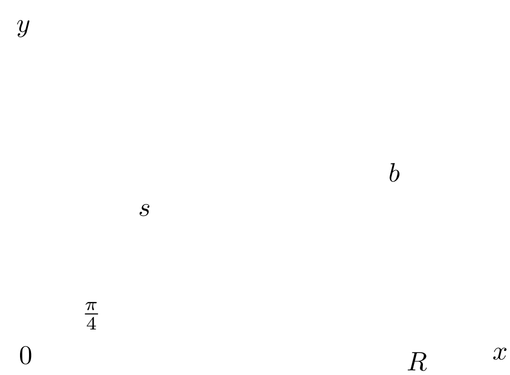
$$\int_0^\infty \cos x^2\,dx \;=\; \int_0^\infty \sin x^2\,dx \;=\; \frac{\sqrt{2\pi}}{4}$$

Proof.



The function $z \mapsto e^{-z^2}$ is entire, whence by the fundamental theorem of complex analysis we have

$$\oint_\gamma e^{-z^2}\,dz \;=\; 0 \tag{1}$$

where γ is the perimeter of the circular sector described in the picture. We split this contour integral to three portions:

$$\underbrace{\int_0^R e^{-x^2}\,dx}_{I_1} + \underbrace{\int_b e^{-z^2}\,dz}_{I_2} + \underbrace{\int_s e^{-z^2}\,dz}_{I_3} \;=\; 0 \tag{2}$$

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By the entry concerning the Gaussian integral, we know that

$$\lim_{R\to\infty} I_1 = \frac{\sqrt{\pi}}{2}.$$

For handling I_2, we use the substitution

$$z := Re^{i\varphi} = R(\cos\varphi + i\sin\varphi), \quad dz = iRe^{i\varphi}\,d\varphi \quad (0 \leqq \varphi \leqq \frac{\pi}{4}).$$

Using also de Moivre's formula we can write

$$|I_2| = \left|iR\int_0^{\frac{\pi}{4}} e^{-R^2(\cos 2\varphi + i\sin 2\varphi)} e^{i\varphi} d\varphi\right| \leqq R\int_0^{\frac{\pi}{4}} \left|e^{-R^2(\cos 2\varphi + i\sin 2\varphi)}\right| \cdot \left|e^{i\varphi}\right| \cdot |d\varphi| = R\int_0^{\frac{\pi}{4}} e^{-R^2\cos 2\varphi} d\varphi.$$

Comparing the graph of the function $\varphi \mapsto \cos 2\varphi$ with the line through the points $(0,\,1)$ and $(\frac{\pi}{4},\,0)$ allows us to estimate $\cos 2\varphi$ downwards:

$$\cos 2\varphi \geqq 1 - \frac{4\varphi}{\pi} \quad \text{for} \quad 0 \leqq \varphi \leqq \frac{\pi}{4}$$

Hence we obtain

$$|I_2| \leqq R\int_0^{\frac{\pi}{4}} \frac{d\varphi}{e^{R^2\cos 2\varphi}} \leqq R\int_0^{\frac{\pi}{4}} \frac{d\varphi}{e^{R^2(1-\frac{4\varphi}{\pi})}} \leqq \frac{R}{e^{R^2}}\int_0^{\frac{\pi}{4}} e^{\frac{4R^2}{\pi}\varphi} d\varphi,$$

and moreover

$$|I_2| \leqq \frac{\pi}{4Re^{R^2}}(e^{R^2} - 1) < \frac{\pi e^{R^2}}{4Re^{R^2}} = \frac{\pi}{4R} \to 0 \quad \text{as} \quad R \to \infty.$$

Therefore

$$\lim_{R\to\infty} I_2 = 0.$$

Then make to I_3 the substitution

$$z := \frac{1+i}{\sqrt{2}}t, \quad dz = \frac{1+i}{\sqrt{2}}dt \quad (R \geqq t \geqq 0).$$

It yields

$$\begin{aligned} I_3 &= \frac{1+i}{\sqrt{2}}\int_R^0 e^{-it^2}\,dt = -\frac{1}{\sqrt{2}}\int_0^R (1+i)(\cos t^2 - i\sin t^2)\,dt \\ &= -\frac{1}{\sqrt{2}}\left(\int_0^R \sin t^2\,dt + \int_0^R \cos t^2\,dt\right) + \frac{i}{\sqrt{2}}\left(\int_0^R \sin t^2\,dt - \int_0^R \cos t^2\,dt\right). \end{aligned}$$

Thus, letting $R \to \infty$, the equation (2) implies

$$\frac{\sqrt{\pi}}{2} + 0 - \frac{1}{\sqrt{2}}\left(\int_0^\infty \sin t^2\,dt + \int_0^\infty \cos t^2\,dt\right) + \frac{i}{\sqrt{2}}\left(\int_0^\infty \sin t^2\,dt - \int_0^\infty \cos t^2\,dt\right) = 0. \tag{3}$$

Because the imaginary part vanishes, we infer that $\int_0^\infty \cos x^2\, dx = \int_0^\infty \sin x^2\, dx$, whence (3) reads

$$\frac{\sqrt{\pi}}{2} + 0 - \frac{1}{\sqrt{2}} \cdot 2 \int_0^\infty \sin t^2\, dt \;=\; 0.$$

So we get also the result $\int_0^\infty \sin x^2\, dx = \frac{\sqrt{2}}{2} \cdot \frac{\sqrt{\pi}}{2} = \frac{\sqrt{2\pi}}{4}$, Q.E.D.