groupoid representation theorem*

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0.1 Groupoid representation theorem

We shall briefly consider a main result due to Hahn (1978) that relates groupoid and associated groupoid algebra representations:

Theorem 0.1. (source: [?, ?].) *Any representation of a groupoid* G *with Haar measure* (ν, μ) *in a separable Hilbert space* $\mathcal{H}$ *induces a *-algebra representation* $f \mapsto X_f$ *of the associated groupoid algebra* $\Pi\mathsf{G}, \nu)$ *in* $L^2(U_{\mathsf{G}}, \mu, \mathcal{H})$ *with the following properties:*

(1) For any $l, m \in H$ *, one has that* $|< X_f(u \mapsto l), (u \mapsto m) >| \leq \|f_l\| \, \|l\| \, \|m\|$ *and*

(2) $M_r(\alpha)X_f = X_{f\alpha\circ r}$*, where* $M_r : L^\infty(U_{\mathsf{G}}, \mu, \mathcal{H}) \longrightarrow \mathcal{L}(L^2(U_{\mathsf{G}}, \mu, \mathcal{H}))$*, with* $M_r(\alpha)j = \alpha \cdot j$*.*

*Conversely, any *-algebra representation with the above two properties induces a groupoid representation, X, as follows:*

$$\langle X_f, j, k\rangle = \int f(x)[X(x)j(d(x)), k(r(x))d\nu(x)], \tag{0.1}$$

(cf. p. 50 of Hahn, 1978).

0.2 Remarks

Furthermore, according to Seda (1986, on p.116) the continuity of a Haar system is equivalent to the continuity of the convolution product $f * g$ for any pair f, g of continuous functions with compact support. One may thus conjecture that similar results could be obtained for functions with *locally compact* support in dealing with convolution products of either locally compact groupoids or quantum groupoids. Seda's result also implies that the convolution algebra $C_{conv}(\mathsf{G})$ of a groupoid G is closed with respect to the convolution * if and only if the fixed Haar system associated with the measured groupoid G is *continuous* (Buneci, 2003).

In the case of groupoid algebras of transitive groupoids, Buneci (2003) showed that representations of a measured groupoid $(\mathsf{G}, [\int \nu^u d\tilde{\lambda}(u)] = [\lambda])$ on a separable Hilbert space $\mathcal{H}$ induce *non-degenerate* $*$–representations $f \mapsto X_f$ of the associated groupoid algebra $\Pi(\mathsf{G}, \nu, \tilde{\lambda})$ with properties formally similar to (1) and (2) above. Moreover, as in the case of groups, *there is a correspondence between the unitary representations of a groupoid and its associated C*–convolution algebra representations* (p.182 of Buneci, 2003), the latter involving however fiber bundles of Hilbert spaces instead of single Hilbert spaces. Therefore, groupoid representations appear as the natural construct for algebraic quantum field theories (AQFT) in which nets of local observable operators in Hilbert space fiber bundles were introduced by Rovelli (1998).

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