functor category*

CWoo[†]

2013-03-22 1:47:14

Let $\mathcal{C},\mathcal{D}$ be categories. Consider the class O of all covariant functors $T:\mathcal{C}\to\mathcal{D}$, and the class M of all natural transformations $\tau:S\dot{\to}T$ for every pair $S,T:\mathcal{C}\to\mathcal{D}$ of functors. Write $\mathcal{D}^{\mathcal{C}}$ for the pair (O,M).

For each pair of functors $S,T:\mathcal{C}\to\mathcal{D}$, write $\hom(S,T)$ the class of all natural transformations from S to T. If τ is in both $\hom(S,T)$ and $\hom(U,V)$, then $S=U$ and $T=V$.

Using the composition of natural transformations, we have a mapping

$$\bullet:\hom(R,S)\times\hom(S,T)\to\hom(R,T),$$

for every triple $R,S,T:\mathcal{C}\to\mathcal{D}$. Since composition of natural transformations is associative, the associativity of $\bullet$ applies.

In addition, for each $S:\mathcal{C}\to\mathcal{D}$, we have the identity natural transformation $1_S\in\hom(S,S)$. For every $\tau\in\hom(S,T)$ and every $\eta\in\hom(T,S)$, we have $\tau\bullet 1_S=\tau$ and $1_S\bullet\eta=\eta$.

From the discussion above, we are ready to call $\mathcal{D}^{\mathcal{C}}$ a category. However, unless $\hom(S,T)$ is a set for every pair of functors in O, $\mathcal{D}^{\mathcal{C}}$ is not a category. When $\mathcal{D}^{\mathcal{C}}$ is a category, we call it the *category of functors* from $\mathcal{C}$ to $\mathcal{D}$, or simply a *functor category*.

That $\mathcal{D}^{\mathcal{C}}$ is a functor category depends on various restrictions being placed on the "sizes" of $\mathcal{C}$ and $\mathcal{D}$:

Proposition 1. *If $\mathcal{C}$ is $\mathcal{U}$-small, then $\mathcal{D}^{\mathcal{C}}$ is a category.*

Proof. Suppose $\mathcal{C}$ is $\mathcal{U}$-small. Consider the class $\hom(S,T)$. Each $\tau\in\hom(S,T)$ is determined by the collection of morphisms $S(A)\to T(A)$ for each object A in $\mathcal{C}$. This means that, for each A in $\mathcal{C}$, $\hom(S(A),T(A))$ contains the image of every $\tau\in\hom(S,T)$ under A. So the class of all these natural transformations is a subclass of the product

$$\prod_{A\in\mathrm{Ob}(\mathcal{C})}\hom(S(A),T(A)) \tag{1}$$

*⟨*FunctorCategory*⟩ created: ⟨*2013-03-2*⟩ by: ⟨*CWoo*⟩ version: ⟨*41082*⟩ Privacy setting: ⟨*1*⟩ ⟨*Definition*⟩ ⟨*18-00*⟩ ⟨*18A25*⟩ ⟨*18A05*⟩

Since $\mathrm{Ob}(\mathcal{C})$, as well as each $\hom(S(A), T(A))$ is a set, so is the product (1). Hence $\hom(S, T)$, being a subclass of (1), is a set, or that $\mathcal{D}^{\mathcal{C}}$ is a category. $\square$

Proposition 2. *If in addition $\mathcal{D}$ is a $\mathcal{U}$-category, then so is $\mathcal{D}^{\mathcal{C}}$.*

Proof. $\mathcal{D}$ being a $\mathcal{U}$-category means that $\hom(S(A), T(A))$ is $\mathcal{U}$-small, for every object A in $\mathcal{C}$. Since $\mathrm{Ob}(\mathcal{C})$ is also $\mathcal{U}$-small (assumption in Proposition 1), the product (1) above is $\mathcal{U}$-small. Consequently, $\hom(S, T)$, being a subclass of (1), is $\mathcal{U}$-small. This shows that $\mathcal{D}^{\mathcal{C}}$ is a $\mathcal{U}$-category. $\square$

Proposition 3. *If $\mathcal{D}$ is furthermore $\mathcal{U}$-small, so is $\mathcal{D}^{\mathcal{C}}$.*

Proof. We want to show that the class $\mathcal{M}$ of all functors from $\mathcal{C}$ to $\mathcal{D}$ is $\mathcal{U}$-small. A functor $S : \mathcal{C} \to \mathcal{D}$ can be broken up into two components: a function $S_1 : \mathrm{Ob}(\mathcal{C}) \to \mathrm{Ob}(\mathcal{D})$, and a function $S_2 : \mathrm{Mor}(\mathcal{C}) \to \mathrm{Mor}(\mathcal{D})$, so that $S_2(A \to B) = S_1(A) \to S_1(B)$.

Define a binary relation $\sim$ on $\mathcal{M}$ so that $S \sim T$ iff they have the same first component: $S_1 = T_1$. It is easy to see that $\sim$ is an equivalence relation on $\mathcal{M}$. Let $[S]$ be the equivalence class containing the functor S. For every morphism $A \to B$, its image under the second component of every functor in $[S]$ lies in $\hom(S_1(A), S_1(B))$. So the size of $[S]$ can not exceed the size of

$$\prod_{A,B \in \mathrm{Ob}(\mathcal{C})} \hom(S_1(A), S_1(B))$$

Since $\mathrm{Ob}(\mathcal{C})$ is $\mathcal{U}$-small (assumption in Prop 1), so is $\mathrm{Ob}(\mathcal{C}) \times \mathrm{Ob}(\mathcal{C})$. Furthermore, since each $\hom(S_1(A), S_1(B))$ is $\mathcal{U}$-small (assumption in Prop 2), $[S]$ is $\mathcal{U}$-small as well.

Next, let us estimate the size of the class $\mathcal{M}/\sim$ of equivalence classes in $\mathcal{M}$. First, note that for every functor $S : \mathcal{C} \to \mathcal{D}$, its first component is a function from the *set* $\mathrm{Ob}(\mathcal{C})$ to the *set* $\mathrm{Ob}(\mathcal{D})$ by assumption. As $[S] \neq [T]$ iff $S_1 \neq T_1$, the size can not exceed

$$|\,\mathrm{Ob}(\mathcal{D})^{\mathrm{Ob}(\mathcal{C})}|$$

the cardinality of the set of all functions from $\mathrm{Ob}(\mathcal{C})$ to $\mathrm{Ob}(\mathcal{D})$. By assumption, $\mathrm{Ob}(\mathcal{D})$ is $\mathcal{U}$-small, so is $\mathrm{Ob}(\mathcal{D})^{\mathrm{Ob}(\mathcal{C})}$. As a result, $\mathcal{M}/\sim$ is $\mathcal{U}$-small. Together with the fact that $[S]$ is $\mathcal{U}$-small for each functor S, we have that $\mathcal{M}$ itself must be $\mathcal{U}$-small, which completes the proof. $\square$