compositions of natural transformations*

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The usual way to compose two natural transformation is what is known as the *vertical composition*. Given categories $\mathcal{C},\mathcal{D}$, functors R,S,T from $\mathcal{C}$ to $\mathcal{D}$, and natural transformations $\tau:R\Rightarrow S$ and $\eta:S\Rightarrow T$, we have a natural transformation $\eta\bullet\tau:R\Rightarrow T$ given by

$$(\eta\bullet\tau)_A:=\eta_A\circ\tau_A$$

The reason for calling $\bullet$ the "vertical" composition is illustrated by the diagram below:

$$@+=3cm\mathcal{C}[0,1][0,1]\mathcal{D} \quad := \quad @+=3cm\mathcal{C}[0,1][r]|S[0,1]\mathcal{D}$$

However, there is another way to compose natural transformations: the so-called *horizontal composition*. Given categories, $\mathcal{B},\mathcal{C},\mathcal{D}$, functors $S_1,T_1:\mathcal{B}\to\mathcal{C}$, $S_2,T_2:\mathcal{C}\to\mathcal{D}$, and natural transformations, $\tau:S_1\Rightarrow T_1$ and $\eta:S_2\Rightarrow T_2$ as in the following diagram

$$@+=3cm\mathcal{B}[0,1][0,1]\mathcal{D} \quad := \quad @+=3cm\mathcal{B}[0,1][0,1]\mathcal{C}[0,1][0,1]\mathcal{D}$$

we define the *horizontal composition*, or *Godemont product*, of η and τ, written $\eta\circ\tau:S_2S_1\to T_2T_1$, as follows: first, pick any object A in $\mathcal{B}$. Because η is a natural transformation, we have a commutative diagram (solid arrows) below

$$@+=2cmS_2S_1(A)[r]^{\eta_{S_1(A)}}[d]_{S_2(\tau_A)}@-->[dr]^{(\eta\circ\tau)_A}T_2S_1(A)[d]^{T_2(\tau_A)}S_2T_1(A)[r]_{\eta_{T_1(A)}}T_2T_1(A)$$

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From this, we set $(\eta \circ \tau)_A$ to be the "diagonal" morphism (dotted arrow) from $S_2S_1(A)$ to $T_2T_1(A)$ in the diagram above:

$$(\eta \circ \tau)_A := T_2(\tau_A) \circ \eta_{S_1(A)} = \eta_{T_1(A)} \circ S_2(\tau_A).$$

Below are some properties of $\circ$:

1. $\eta \circ \tau$ is a natural transformation.
2. $\circ$ is associative.
3. $\eta \circ 1_S = \eta$, and $1_S \circ \tau = \tau$, where η and τ are described in the diagram above, and 1_S is the identity transformation on the functor $S : \mathcal{B} \to \mathcal{C}$, and 1_T is the identity transformation on the functor $T : \mathcal{C} \to \mathcal{D}$.
4. $\circ$ and $\bullet$ satisfy the interchange law.

In fact, the first three properties above turn **Cat**, the category of small categories, into a category where the objects are small categories, morphisms are natural transformations, and the composition of morphisms is the horizontal composition.