# theory for separation of variables*

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The first order ordinary differential equation where one can separate the variables has the form where $\frac{dy}{dx}$ may be expressed as a product or a quotient of two functions, one of which depends only on $x$ and the other on $y$. Such an equation may be written e.g. as

$$\frac{dy}{dx} = \frac{Y(y)}{X(x)} \quad \text{or} \quad \frac{dx}{dy} = \frac{X(x)}{Y(y)}. \tag{1}$$

We notice first that if $Y(y)$ has real zeroes $y_1, y_2, \ldots$, then the equation (1) has the constant solutions $y := y_1,\ y := y_2,\ \ldots$ and thus the lines $y = y_1,\ y = y_2,\ \ldots$ are integral curves. Similarly, if $X(x)$ has real zeroes $x_1, x_2, \ldots$, one has to include the lines $y = y_1,\ y = y_2,\ \ldots$ to the integral curves. All those lines divide the $xy$-plane into the rectangular regions. One can obtain other integral curves only inside such regions where the derivative $\frac{dy}{dx}$ attains real values.

Let $R$ be such a region, defined by

$$a < x < b, \quad c < y < d,$$

and let us assume that the $X(x)$ and $Y(y)$ are real, continuous and distinct from zero in $R$. We will show that any integral curve of the differential equation (1) is accessible by two quadratures.

Let $\gamma$ be an integral curve passing through the point $(x_0,\, y_0)$ of the region $R$. By the above assumptions, the derivative $\frac{dy}{dx}$ maintains its sign on the curve $\gamma$ so long $\gamma$ is inside $R$, which is true on a neighbourhood $N$ of $x_0$, contained in $[a,\, b]$. This implies that as $x$ runs the interval $N$, it defines the ordinate $y$ of $\gamma$ uniquely as a monotonic function $y \mapsto y(x)$ which satisfies the equation (1):

$$y'(x) = \frac{Y(y(x))}{X(x)}$$

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The last equation may be written

$$\frac{y'(x)}{Y(y(x))} = \frac{1}{X(x)}. \tag{2}$$

Since $X$ and $Y$ don't vanish in $R$, the denominators $Y(y(x))$ and $X(x)$ are distinct from 0 on the interval $N$. Therefore one can integrate both sides of (2) from $x_0$ to an arbitrary value $x$ on $N$, getting

$$\int_{x_0}^{x} \frac{y'(x)\,dx}{Y(y(x))} = \int_{x_0}^{x} \frac{dx}{X(x)}. \tag{3}$$

Because $y = y(x)$ is continuous and monotonic on the interval $N$, it can be taken as new variable of integration in the left hand side of (3): substitute $y(x) := y,\ y'(x)\,dx := dy$ and change the limits to $y(x_0) = y_0$ and $y(x) = y$.

- Accordingly, the equality

  $$\int_{y_0}^{y} \frac{dy}{Y(y)} = \int_{x_0}^{x} \frac{dx}{X(x)} \tag{4}$$

  is valid, meaning that if an integral curve of (1) passes through the point $(x_0,\, y_0)$, the integral curve is represented by the equation (4) as long as the curve is inside the region $R$.

- Additionally, it is possible to justificate that if $(x_0,\, y_0)$ is an interior point of a region $R$ where $X(x)$ and $Y(y)$ are real, continuous and $\neq 0$, then one and only one integral curve of (1) passes through this point, the curve is regular, and both $x$ and $y$ are monotonic on it. N.B., the Lipschitz condition for the right hand side of (1) is not necessary for the justification.

- When the point $(x_0,\, y_0)$ changes in the region $R$, (4) gives a family of integral curves which cover the region once. The equations of these curves may be unified to the form

  $$\int \frac{dy}{Y(y)} = \int \frac{dx}{X(x)}, \tag{5}$$

  which thus represents the general solution of the differential equation (1) in $R$. Hence one can speak of the *separation of variables*,

  $$\frac{dy}{Y(y)} = \frac{dx}{X(x)}, \tag{6}$$

  and integration of both sides.

## References

[1] E. Lindelöf: *Differentiali- ja integralilasku III 1.* Mercatorin Kirjapaino Osakeyhtiö, Helsinki (1935).