derivation of heat equation*

pahio†

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Let us consider the heat conduction in a homogeneous matter with density ϱ and specific heat capacity c. Denote by $u(x,\,y,\,z,\,t)$ the temperature in the point $(x,\,y,\,z)$ at the time t. Let a be a simple closed surface in the matter and v the spatial region restricted by it.

When the growth of the temperature of a volume element dv in the time dt is du, the element releases the amount

$$-du\;c\,\varrho\,dv \;=\; -u'_t\,dt\,c\,\varrho\,dv$$

of heat, which is the heat flux through the surface of dv. Thus if there are no sources and sinks of heat in v, the heat flux through the surface a in dt is

$$-dt\int_v c\varrho u'_t\,dv. \tag{1}$$

On the other hand, the flux through da in the time dt must be proportional to a, to dt and to the derivative of the temperature in the direction of the normal line of the surface element da, i.e. the flux is

$$-k\,\nabla u\cdot d\vec{a}\;dt,$$

where k is a positive constant (because the heat flows always from higher temperature to lower one). Consequently, the heat flux through the whole surface a is

$$-dt\oint_a k\nabla u\cdot d\vec{a},$$

which is, by the Gauss's theorem, same as

$$-dt\int_v k\,\nabla\cdot\nabla u\,dv \;=\; -dt\int_v k\,\nabla^2 u\,dv. \tag{2}$$

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Equating the expressions (1) and (2) and dividing by dt, one obtains

$$\int_v k\,\nabla^2 u\,dv \;=\; \int_v c\,\varrho u'_t\,dv.$$

Since this equation is valid for any region v in the matter, we infer that

$$k\,\nabla^2 u \;=\; c\,\varrho u'_t.$$

Denoting $\dfrac{k}{c\varrho} = \alpha^2,$ we can write this equation as

$$\alpha^2\nabla^2 u \;=\; \frac{\partial u}{\partial t}. \tag{3}$$

This is the differential equation of heat conduction, first derived by Fourier.

References

[1] K. Väisälä: *Matematiikka IV.* Handout Nr. 141. Teknillisen korkeakoulun ylioppilaskunta, Otaniemi, Finland (1967).