pencil of conics*

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Two conics

$$U = 0 \quad \text{and} \quad V = 0 \tag{1}$$

can intersect in four points, some of which may coincide or be "imaginary".

The equation

$$pU + qV = 0, \tag{2}$$

where p and q are freely chooseable parametres, not both 0, represents the *pencil* of all the conics which pass through the four intersection points of the conics (1); see quadratic curves.

The same pencil is gotten by replacing one of the conics (1) by two lines $L_1 = 0$ and $L_2 = 0$, such that the first line passes through two of the intersection points and the second line through the other two of those points; then the equation of the pencil reads

$$pL_1L_2 + qV = 0. \tag{3}$$

One can also replace similarly the other (V) of the conics (1) by two lines $L_3 = 0$ and $L_4 = 0$; then the pencil of conics is

$$pL_1L_2 + qL_3L_4 = 0. \tag{4}$$

For any pair $(p,\,q)$ of values, one conic section (4) passes through the four points determined by the equation pairs

$$L_1 = 0 \,\wedge\, L_3 = 0, \quad L_1 = 0 \,\wedge\, L_4 = 0, \quad L_2 = 0 \,\wedge\, L_3 = 0, \quad L_2 = 0 \,\wedge\, L_4 = 0.$$

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The pencils given by the equations (2), (3) and (4) can be obtained also by fixing either of the parametres p and q for example to -1, when e.g. the pencil (4) may be expressed by

$$pL_1L_2 \;=\; L_3L_4. \tag{5}$$

Application. Using (5), we can easily find the equation of a conics which passes through five given points; we may first form the equations of the sides $L_1 = 0$, $L_2 = 0$, $L_3 = 0$ and $L_4 = 0$ of the quadrilateral determined by four of the given points. The equation of the searched conic is then (5), where the value of p is gotten by substituting the coordinates of the fifth point to (5) and by solving p.

Example. Find the equation of the conic section passing through the points

$$(-1,\,0),\quad (1,\,0),\quad (0,\,1),\quad (0,\,2),\quad (2,\,2).$$

We can take the lines

$$2x+y-2=0,\quad x-y+1=0,\quad 2x-y+2=0,\quad x+y-1=0$$

passing through pairs of the four first points. The equation of the pencil of the conics passing through these points is thus of the form

$$p(2x+y-2)(x-y+1) \;=\; (2x-y+2)(x+y-1). \tag{6}$$

The conics passes through $(2,\,2)$, if we substitute $x := 2$, $y := 2$; it follows that $p = 3$. Using this value in (6) results the equation of the searched conics:

$$2x^2-y^2-2xy+3y-2 \;=\; 0 \tag{7}$$

The coefficients 2, -1, -2 of the second degree terms let infer, that this curve is a hyperbola with axes not parallel to the coordinate axes (see quadratic curves).