

truth-value semantics for classical propositional logic*

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In classical propositional logic, an *interpretation* of a well-formed formula (wff) p is an assignment of truth (=1) or falsity (=0) to p . Any interpreted wff is called a proposition.

An *interpretation* of all wffs over the variable set V is then a Boolean function on \overline{V} . However, one needs to be careful, for we do not want both p and $\neg p$ be interpreted as true simultaneously (at least not in classical propositional logic)! The proper way to find an interpretation on the wffs is to start from the atoms.

Call any Boolean-valued function ν on V a *valuation* on V . We want to extend ν to a Boolean-valued function $\overline{\nu}$ on \overline{V} of all wffs. The way this is done is similar to the construction of wffs; we build a sequence of functions, starting from ν on V_0 , next ν_1 on V_1 , and so on... Finally, we take the union of all these “approximations” to arrive at $\overline{\nu}$. So how do we go from ν to ν_1 ? We need to interpret $\neg p$ and $p \vee q$ from the valuations of atoms p and q . In other words, we must also interpret logical connectives too.

Before doing this, we define a truth function for each of the logical connectives:

- for \neg , define $f : \{0, 1\} \rightarrow \{0, 1\}$ given by $f(x) = 1 - x$.
- for \vee , define $g : \{0, 1\}^2 \rightarrow \{0, 1\}$ given by $g(x, y) = \max(x, y)$.

As we are trying to *interpret* \neg (not) and \vee (or), the choices for f and g are clear. The values 0, 1 are interpreted as the usual integers (so they can be subtracted and ordered, etc...). Hence f and g make sense.

Next, recall that V_i are sets of wffs built up from wffs in V_{i-1} (see construction of well-formed formulas for more detail). We define a function $\nu_i : V_i \rightarrow \{0, 1\}$ for each i , as follows:

- $\nu_0 := \nu$

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- suppose ν_i has been defined, we define $\nu_{i+1} : V_{i+1} \rightarrow \{0, 1\}$ given by

$$\nu_{i+1}(p) := \begin{cases} \nu_i(p) & \text{if } p \in V_i, \\ f(\nu_i(q)) & \text{if } p = \neg q \text{ for some } q \in V_i, \\ g(\nu_i(q), \nu_i(r)) & \text{if } p = q \vee r \text{ for some } q, r \in V_i. \end{cases}$$

Finally, take $\bar{\nu}$ to be the union of all the approximations:

$$\bar{\nu} := \bigcup_{i=0}^{\infty} \nu_i.$$

Then, by unique readability of wffs, $\bar{\nu}$ is an interpretation on \bar{V} .

Remark. If \perp is included in the language of the logic (as the symbol for falsity), we also require that $\nu_i(\perp) = \bar{\nu}(\perp) = 0$.

Remark \bar{V} can be viewed as an inductive set over V with respect to the \neg and \vee , viewed as operations on \bar{V} . Furthermore, \bar{V} is freely generated by V , since each V_{i+1} can be partitioned into sets V_i , $\{(p \vee q) \mid p, q \in V_i\}$, and $\{(\neg p) \mid p \in V_i\}$, and each partition is non-empty. As a result, any valuation ν on V uniquely extends to a valuation $\bar{\nu}$ on \bar{V} .

Definitions. Let p, q be wffs in \bar{V} .

- p is *true* or *satisfiable* for some valuation ν if $\bar{\nu}(p) = 1$ (otherwise, it is *false* for ν).
- p is true for every valuation ν , then p is said to be *valid* (or *tautologous*). If p is false for every ν , it is *invalid*. If p is valid, we write $\models p$.
- p implies q for a valuation ν if $\bar{\nu}(p) = 1$ implies $\bar{\nu}(q) = 1$. p *semantically implies* if p implies q for every valuation ν , and is denoted by $p \models q$.
- p is *equivalent* to q for ν if $\bar{\nu}(p) = \bar{\nu}(q)$. They are *semantically equivalent* if they are equivalent for every ν , and written $p \equiv q$.

Semantical equivalence is an equivalence relation on \bar{V} .

The above can be easily generalized to sets of wffs. Let T be a set of propositions.

- T is true or satisfiable for ν if $\bar{\nu}(T) = \{1\}$ (otherwise, it is false for ν).
- T is valid if it is true for every ν ; it is *invalid* if it is false for every ν . If T is valid, we write $\models T$.
- T implies p for ν if $\bar{\nu}(T) = \{1\}$ implies $\bar{\nu}(p) = 1$. T *semantically implies* p if T implies p for every ν , and is denoted by $T \models p$.
- T_1 implies T_2 for ν if, for every $p \in T_2$, T_1 implies p for ν . T_1 *semantically implies* T_2 if T_1 implies T_2 for every ν , and is denoted by $T_1 \models T_2$.
- T_1 is equivalent to T_2 for ν if for some valuation ν , T_1 implies T_2 for ν and T_2 implies T_1 for ν . T_1 and T_2 are *semantically equivalent* if $T_1 \models T_2$ and $T_2 \models T_1$, written $T_1 \equiv T_2$.

Clearly, $\models p$ iff $\emptyset \models p$, and $T \models p$ iff $T \models \{p\}$.