quotient of languages

CWoo

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Let \( L_1, L_2 \) be two languages over some alphabet \( \Sigma \). The quotient of \( L_1 \) by \( L_2 \) is defined to be the set

\[
L_1 / L_2 := \{ u \in \Sigma^* | uv \in L_1 \text{ for some } v \in L_2 \}.
\]

\( L_1 / L_2 \) is sometimes written \( L_1 L_2^{-1} \). The quotient so defined is also called the right quotient, for one can similarly define the left quotient of \( L_1 \) by \( L_2 \):

\[
L_1 \setminus L_2 := \{ u \in \Sigma^* | vu \in L_1 \text{ for some } v \in L_2 \}.
\]

\( L_1 \setminus L_2 \) is sometimes written \( L_2^{-1} L_1 \).

Below are some examples of quotients:

- If \( L_1 = \{ a^n b^n c^n | n \geq 0 \} \) and \( L_2 = \{ b, c \}^* \), then
  - \( L_1 / L_2 = \{ a^m b^n | m \geq n \geq 0 \} \)
  - \( L_2 / L_1 = L_2 \)
  - \( L_1 \setminus L_2 = \{ \lambda \} \), the singleton consisting the empty word
  - \( L_2 \setminus L_1 = L_2 \)

- for any language \( L \) over \( \Sigma \):
  - \( L / \Sigma^* \) is the language of all prefixes of words of \( L \)
  - \( \Sigma^* / L = \Sigma^* \)
  - \( L / \Sigma^* \) is the language of all suffixes of words of \( L \)
  - \( \Sigma^* \setminus L = \Sigma^* \)

- \( \lambda \in L / L \cap L \setminus L \), and if \( \lambda \in L \), then \( L \subseteq L / L \cap L \setminus L \).

Here are some basic properties of quotients:

1. \( L_1 \subseteq (L_1 / L_2) L_2 \cap L_2 (L_1 \setminus L_2) \).

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2. \((L_1/L_2)L_2 \subseteq (L_1 L_2)/L_2\), and \(L_2(L_1 \backslash L_2) \subseteq (L_2 L_1) \backslash L_2\).

3. Right and left quotients are related via reversal:

\[
\begin{align*}
(L_1 \backslash L_2)^R &= \{u^R \mid vu \in L_1 \text{ for some } v \in L_2\} \\
&= \{u^R \mid (vu)^R \in L_1^R \text{ for some } v^R \in L_2^R\} \\
&= \{u^R \mid u^R v^R \in L_1^R \text{ for some } v^R \in L_2^R\} \\
&= L_1^R / L_2^R.
\end{align*}
\]

A family \(\mathcal{F}\) of languages is said to be \textit{closed under quotient by a language} \(L\) if for every language \(M \in \mathcal{F}\), \(M/L \in \mathcal{F}\). Furthermore, \(\mathcal{F}\) is said to be \textit{closed under quotient} if \(M/L \in \mathcal{F}\) for any \(M, L \in \mathcal{F}\). Closure under quotient is also termed closure under right quotient. Closure under left quotient is similarly defined.

It can be shown that the families of regular, context-free, and type-0 languages are closed under quotient (both left and right) by a regular language. The family of context-sensitive languages does not have this closure property.

Since all of the families mentioned above are closed under reversal, each of the families, except the context-sensitive family, is closed under left quotient by a regular language, according to the second property above.

References