

application of logarithm series*

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The integrand of the improper integral

$$I := \int_0^1 \frac{\ln(1+x)}{x} dx \quad (1)$$

is not defined at the lower limit 0. However, from the Taylor series expansion

$$\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + -\dots \quad (-1 < x \leq 1)$$

of the natural logarithm we obtain the expansion of the integrand

$$\frac{\ln(1+x)}{x} = 1 - \frac{x}{2} + \frac{x^2}{3} - \frac{x^3}{4} + -\dots \quad (-1 < x < 0, 0 < x \leq 1)$$

whence

$$\lim_{x \rightarrow 0} \frac{\ln(1+x)}{x} = 1. \quad (2)$$

This implies that the integrand of (1) is bounded on the interval $[0, 1]$ and also continuous, if we think that (2) defines its value at $x = 0$. Accordingly, the integrand is Riemann integrable on the interval, and we can determine the improper integral by integrating termwise:

$$\begin{aligned} I &= \int_0^1 \left(1 - \frac{x}{2} + \frac{x^2}{3} - \frac{x^3}{4} + -\dots \right) dx \\ &= \int_0^1 \left(x - \frac{x^2}{2^2} + \frac{x^3}{3^2} - \frac{x^4}{4^2} + -\dots \right) \\ &= 1 - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} + -\dots \end{aligned}$$

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By the entry on Dirichlet eta function at 2, the sum of the obtained series is $\eta(2) = \frac{\pi^2}{12}$. Thus we have the result

$$\int_0^1 \frac{\ln(1+x)}{x} dx = \frac{\pi^2}{12}. \quad (3)$$

Similarly, using the series

$$\ln(1-x) = -x - \frac{x^2}{2} - \frac{x^3}{3} - \frac{x^4}{4} - \dots \quad (-1 \leq x < 1)$$

and the result in the entry Riemann zeta function at 2, one can calculate that

$$\int_0^1 \frac{\ln(1-x)}{x} dx = -\frac{\pi^2}{6}. \quad (4)$$