Let $\Sigma$ be an alphabet, and $u, v$ words over $\Sigma$. An insertion of $u$ into $v$ is a word of the form $v_1u_1v_2$, where $v = v_1v_2$. The concatenation of two words is just a special case of insertion. Also, if $w$ is an insertion of $u$ into $v$, then $v$ is a deletion of $u$ from $w$.

The insertion of $u$ into $v$ is the set of all insertions of $v$ into $u$, and is denoted by $v \triangleright u$.

The notion of insertion can be extended to languages. Let $L_1, L_2$ be two languages over $\Sigma$. The insertion of $L_1$ into $L_2$, denoted by $L_1 \triangleright L_2$, is the set of all insertions of words in $L_1$ into words in $L_2$. In other words,

$$L_1 \triangleright L_2 = \bigcup\{u \triangleright v \mid u \in L_1, v \in L_2\}.$$ 

So $u \triangleright v = \{u\} \triangleright \{v\}$.

A language $L$ is said to be insertion closed if $L \triangleright L \subseteq L$. Clearly $\Sigma^*$ is insertion closed, and arbitrary intersection of insertion closed languages is insertion closed. Given a language $L$, the insertion closure of $L$, denoted by $\text{Ins}(L)$, is the smallest insertion closed language containing $L$. It is clear that $\text{Ins}(L)$ exists and is unique.

More to come...

The concept of insertion can be generalized. Instead of the insertion of $u$ into $v$ taking place in one location in $v$, the insertion can take place in several locations, provided that $u$ must also be broken up into pieces so that each individual piece goes into each inserting location. More precisely, given a positive integer $k$, a $k$-insertion of $u$ into $v$ is a word of the form

$$v_1u_1 \cdots v_ku_kv_{k+1}$$

where $u = u_1 \cdots u_k$ and $v = v_1 \cdots v_{k+1}$. So an insertion is just a 1-insertion. The $k$-insertion of $u$ into $v$ is the set of all $k$-insertions of $u$ into $v$, and is denoted by $u \triangleright^{[k]} v$. Clearly $\triangleright^{[1]} = \triangleright$.

**Example.** Let $\Sigma = \{a, b\}$, and $u = aba$, $v = bab$, and $w = bababa$. Then
• $w$ is an insertion of $u$ into $v$, as well as an insertion of $v$ into $u$, for $w = vu\lambda = \lambda vu$.

• $w$ is also a 2-insertion of $u$ into $v$:
  
  – the decompositions $u = (ab)(a)$ and $v = (b)(ab)\lambda$
  
  – or the decompositions $u = \lambda u$ and $v = \lambda v\lambda$

  produce $(b)(ab)(ab)\lambda = \lambda vu\lambda = w$.

• Finally, $w$ is also a 2-insertion of $v$ into $u$, as the decompositions $u = \lambda u\lambda$ and $v = v\lambda$ produce $\lambda vu\lambda\lambda = w$.

• $u \triangleright v = \{abab, babaab, baabab, bababa\}$.

From this example, we observe that a $k$-insertion is a $(k + 1)$-insertion, and every $k$-insertion of $u$ into $v$ is a $(k + 1)$-insertion of $v$ into $u$. As a result,

$$u \triangleright [k] v \subseteq (u \triangleright [k+1] v) \cap (v \triangleright [k+1] u).$$

As before, given languages $L_1$ and $L_2$, the $k$-insertion of $L_1$ into $L_2$, denoted by $L_1 \triangleright [k] L_2$, is the union of all $u \triangleright [k] v$, where $u \in L_1$ and $v \in L_2$.

**Remark.** Some closure properties regarding insertions: let $\mathcal{R}$ be the family of regular languages, and $\mathcal{F}$ the family of context-free languages. Then $\mathcal{R}$ is closed under $\triangleright[k]$, for each positive integer $k$. $\mathcal{F}$ is closed $\triangleright[k]$ only when $k = 1$. If $L_1 \in \mathcal{R}$ and $L_2 \in \mathcal{F}$, then $L_1 \triangleright[k] L_2$ and $L_2 \triangleright[k] L_1$ are both in $\mathcal{F}$. The proofs of these closure properties can be found in the reference.

**References**