logarithmic spiral*

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2013-03-22 3:05:59

The equation of the *logarithmic spiral* in polar coordinates r, φ is

$$r = Ce^{k\varphi} \tag{1}$$

where C and k are constants ($C>0$). Thus the position vector of the point of this curve as the coordinate vector is written as

$$\vec{r} = (Ce^{k\varphi}\cos\varphi,\ Ce^{k\varphi}\sin\varphi)$$

which is a parametric form of the curve.

Perhaps the most known characteristic of the logarithmic spiral is that any line emanating from the origin cuts the curve under a constant angle ψ. This is seen e.g. by using the vector $\vec{r}$ and its derivative $\frac{d\vec{r}}{d\varphi} = \vec{r}\,'$, the latter of which gives the direction of the tangent line (see vector-valued function):

$$\vec{r}\,' = \left(Ce^{k\varphi}k\cos\varphi - Ce^{k\varphi}\sin\varphi,\ Ce^{k\varphi}k\sin\varphi + Ce^{k\varphi}\cos\varphi\right).$$

One obtains

$$\vec{r}\cdot\vec{r}\,' = kr^2, \quad |\vec{r}| = r, \quad |\vec{r}\,'| = r\sqrt{1+k^2},$$

whence

$$\cos\psi = \frac{\vec{r}\cdot\vec{r}\,'}{|\vec{r}||\vec{r}\,'|} = \frac{k}{\sqrt{1+k^2}} = \text{constant}.$$

It follows that $k = \cot\psi$. The angle ψ is called the polar tangential angle.

The logarithmic spiral (1) goes infinitely many times round the origin without to reach it; in the case $k>0$ one may state that

$$\lim_{\varphi\to-\infty} Ce^{k\varphi} = 0 \quad \text{but} \quad Ce^{k\varphi} \neq 0 \ \forall\varphi\in\mathbb{R}$$

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*⟨*LogarithmicSpiral*⟩ created: ⟨*2013-03-2*⟩ by: ⟨*pahio*⟩ version: ⟨*41917*⟩ Privacy setting: ⟨*1*⟩ ⟨*Topic*⟩ ⟨*14H45*⟩

(the exponential function never vanishes).

The arc length s of the logarithmic spiral is expressible in closed form; if we take it for the interval $[\varphi_1,\,\varphi_2]$, we can calculate in the case $k > 0$ that

$$s \;=\; \int_{\varphi_1}^{\varphi_2}\sqrt{r^2+\left(\frac{dr}{d\varphi}\right)^2}\,d\varphi \;=\; \int_{\varphi_1}^{\varphi_2}\sqrt{C^2e^{2k\varphi}+C^2e^{2k\varphi}k^2}\,d\varphi \;=\; \frac{\sqrt{1+k^2}}{k}C(e^{k\varphi_2}-e^{k\varphi_1}),$$

thus

$$s \;=\; \frac{\sqrt{1+k^2}}{k}(r_2-r_1) \;=\; \frac{r_2-r_1}{\cos\psi}.$$

Letting $\varphi_1 \to -\infty$ one sees that the arc length from the origin to a point of the spiral is finite.

Other properties

- Any curve with constant polar tangential angle is a logarithmic spiral.
- All logarithmic spirals with equal polar tangential angle are similar.
- A logarithmic spiral rotated about the origin is a spiral homothetic to the original one.
- The inversion $z \mapsto \frac{1}{z}$ causes for the logarithmic spiral a reflexion against the imaginary axis and a rotation around the origin, but the image is congruent to the original one.
- The evolute of the logarithmic spiral is a congruent logarithmic spiral.
- The catacaustic of the logarithmic spiral is a logarithmic spiral.
- The families $r = C_1e^{\varphi}$ and $r = C_2e^{-\varphi}$ are orthogonal curves to each other.