

convergent series where not only a_n but also na_n tends to 0*

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Proposition. If the terms a_n of the convergent series

$$a_1 + a_2 + \dots$$

are positive and form a monotonically decreasing sequence, then

$$\lim_{n \rightarrow \infty} na_n = 0. \quad (1)$$

Proof. Let ε be any positive number. By the Cauchy criterion for convergence and the positivity of the terms, there is a positive integer m such that

$$0 < a_{m+1} + \dots + a_{m+p} < \frac{\varepsilon}{2} \quad (p = 1, 2, \dots).$$

Since the sequence a_1, a_2, \dots is decreasing, this implies

$$0 < pa_{m+p} < \frac{\varepsilon}{2} \quad (p = 1, 2, \dots). \quad (2)$$

Choosing here especially $p := m$, we get

$$0 < ma_{m+m} < \frac{\varepsilon}{2},$$

whence again due to the decrease,

$$0 < ma_{m+p} < \frac{\varepsilon}{2} \quad (p = m, m+1, \dots). \quad (3)$$

Adding the inequalities (2) and (3) with the common values $p = m, m+1, \dots$ then yields

$$0 < (m+p)a_{m+p} < \varepsilon \quad \text{for } p \geq m.$$

**(ConvergentSeriesWhereNotOnlyanButAlsoNanTendsTo0)* created: *(2013-03-2)* by: *(pahio)* version: *(41941)* Privacy setting: *(1)* *(Theorem)* *(40A05)*

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This may be written also in the form

$$0 < na_n < \varepsilon \quad \text{for } n \geq 2m$$

which means that $\lim_{n \rightarrow \infty} na_n = 0$.

Remark. The assumption of monotonicity in the Proposition is essential. I.e., without it, one cannot generally get the limit result (1). A counterexample would be the series $a_1 + a_2 + \dots$ where $a_n := \frac{1}{n}$ for any perfect square n but 0 for other values of n . Then this series is convergent (cf. the over-harmonic series), but $na_n = 1$ for each perfect square n ; so $na_n \not\rightarrow 0$ as $n \rightarrow \infty$.