limit points of uncountable subsets of $\mathbb{R}^{n^*}$

$joking^\dagger$

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**Proposition.** Let $\mathbb{R}^n$ be an $n$-dimensional, real normed space and let $A \subseteq \mathbb{R}^n$. If $A$ is uncountable, then there exists limit point of $A$ in $\mathbb{R}^n$.

**Proof.** For any $k \in \mathbb{N}$ let

$$B_k = \{ v \in \mathbb{R}^n \mid ||v|| \leq k \},$$

i.e. $B_k$ is a closed ball centered in 0 with radius $k$. Assume, that for any $k$ the set

$$V_k = B_k \cap A$$

is finite. Then $\bigcup V_k = A$ would be at most countable. Contradiction, since $A$ is uncountable. Thus, there exists $k_0 \in \mathbb{N}$ such that $V_{k_0}$ is infinite. But $V_{k_0} \subseteq B_{k_0}$ and since $B_{k_0}$ is compact (and $V_{k_0}$ is infinite), then there exists limit point of $V_{k_0}$ in $\mathbb{R}^n$. This completes the proof. $\Box$

**Corollary.** If $A \subseteq \mathbb{R}^n$ is uncountable, then there exist infinitely many limit points of $A$ in $\mathbb{R}^n$.

**Proof.** Assume, that there are finitely many limit points of $A$, namely $x_1, \ldots, x_k \in \mathbb{R}^n$. For $\varepsilon > 0$ define

$$A_\varepsilon = \{ v \in \mathbb{R}^n \mid \forall i \, ||v - x_i|| > \varepsilon \}.$$

Briefly speaking, $A_\varepsilon$ is a complement of a union of closed balls centered at $x_i$ with radii $\varepsilon$. Of course $A_\varepsilon \neq \emptyset$ since there are finitely many limit points. Let

$$V_\varepsilon = A \cap A_\varepsilon.$$

Assume, that $V_\varepsilon$ is countable for every $\varepsilon$. Then

$$A \subseteq \bigcup_{n \in \mathbb{N}} V_{1/n} \cup \{ x_1, \ldots, x_k \}$$

would be at most countable (of course under assumption of Axiom of Choice). Contradiction. Thus, there is $\gamma > 0$ such that $V_\gamma$ is uncountable. Then (due to proposition) there is a limit point $x' \in \mathbb{R}^n$ of $V_\gamma$. Note, that

$$x' \in \overline{V_\gamma} \subseteq V_\gamma.'$$

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for some $0 < \gamma' < \gamma$. Thus $x'$ is different from any $x_i$. Contradiction, since $x'$ is also a limit point of $A$. □